182. On the Spherical Derivative of Functions Regular or Meromorphic in the Unit Disc

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1. **Introduction.** O. Lehto and K. Virtanen [3] used the spherical derivative

$$\rho(f(z))=\frac{|f'(z)|}{1+|f(z)|^2} \tag{1.1}$$

as a measure of the growth of $f(z)$ near an isolated singularity, and they [1, 2] developed the study of this direction. In particular, as regards the growth of the spherical derivative Lehto proved:

Theorem A. *Let $f(z)$ be meromorphic in a neighbourhood of the essential singularity $z=a$. Then*

$$\overline{\lim_{z\to a}}\,|z-a|\,\rho(f(z))\geqq\frac{1}{2}. \tag{1.2}$$

Equality holds for the product

$$f(z)=\prod_{\nu}\frac{z-a-a_\nu}{z-a+a_\nu},$$

where the numbers a_ν satisfy the condition $|a_{\nu+1}|=o(|a_\nu|)$.

Theorem B. *If $f(z)$ satisfies the hypothesis of Theorem A and further $f(z)$ is regular near $z=a$, then*

$$\overline{\lim_{z\to a}}\,|z-a|\,\rho(f(z))=\infty. \tag{1.3}$$

Further J. Clunie and W. K. Hayman obtained some extensions of Theorem A and B in their paper [4]. For instance, they proved the following result.

Theorem C. *If $f(z)$ is an integral function of proper order λ $(0\leqq\lambda\leqq\infty)$, then*

$$\overline{\lim_{r\to\infty}}\frac{r\mu(r,f)}{\log M(r,f)}\geqq A_0(\lambda+1), \tag{1.4}$$

where A_0 is an absolute constant and $\mu(r,f)=\sup_{|z|=r}\rho(f(z))$.

2. Our object in this paper is to obtain some results concerning the growth of spherical derivative $\rho(f(z))$ for functions regular and meromorphic in the unit disc $|z|<1$. First we shall prove:

Theorem 1. *Suppose that $f(z)$ is regular for $|z|<1$ and that its order λ satisfies $2<\lambda\leqq\infty$. Then*

$$\overline{\lim_{r\to 1}}\,(1-r)^{\lambda-1}\mu(r,f)\geqq K\lambda\left(\frac{\lambda-2}{\lambda+2}\right)^{\lambda-1} \tag{2.1}$$

holds, where $\mu(r, f)=\sup_{|z|=r} \rho(f(z))$ *and* K *is a positive constant depending on* $f(z)$ *only.*

3. Lemmas. We require two lemmas to prove Theorem 1.

Lemma 1. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ *be regular in* $|z-z_0|\leqq\delta$ *and satisfy* $|f(z)|\geqq 1$ *there. Then*

$$|a_1|\leqq\frac{2|a_0|\log|a_0|}{\delta}. \tag{3.1}$$

If further $|f(z_1)|=1$ *for some* z_1 *with* $|z_1-z_0|=\delta$ *then for some* z *on the segment joining* z_0 *to* z_1

$$\rho(f(z))\geqq\frac{\log|a_0|}{10\delta\log 2}. \tag{3.2}$$

This result was given by W. K. Hayman ([4], p. 125).

Lemma 2. *Suppose that* $\varphi(r)$ $(0<r<1)$ *is continuous, positive and strictly increasing with a piecewise continuous locally bounded derivative* $\varphi'(r)$. [*At points of discontinuity we define* $\varphi'(r)$ *as the limit from the left.*] *Suppose that for positive* α, β

$$\overline{\lim_{r\to 1}}\, \varphi(r)(1-r)^{\alpha}>\beta. \tag{3.3}$$

Then given α' $(0<\alpha'<\alpha)$ *there exist* r *arbitrarily near to* 1 *for which the following are satisfied;*

$$\frac{\varphi'(r)}{\varphi(r)}\geqq\frac{\alpha'}{1-r}. \tag{3.4}$$

$$\varphi(r)(1-r)^{\alpha}\geqq\beta. \tag{3.5}$$

This lemma is an analogue of Hayman's ([4], Lemma 3), so we omit the proof.

4. Proof of Theorem 1. We apply Lemma 2 with $\alpha=\lambda$ and $\alpha>\alpha'>2$ to $\varphi(r)=\log M(r, f)$ so that for some r arbitrarily near to 1, (3.4) and (3.5) hold simultaneously. For such an r there exists a point $z_0=re^{i\theta}$ such that

$$|f(z_0)|=M(r, f), \qquad \left|\frac{f'(z_0)}{f(z_0)}\right|=\varphi'(r). \tag{4.1}$$

(see e.g., [5], p. 136). Now we consider a non-Euclidean disc with the center z_0 and the radius $\delta(r)$

$$D(z_0, \delta(r))=\{z: \sigma(z, z_0)<\delta(r)\}\subset\{|z|<1\}, \tag{4.2}$$

where $\delta(r)$ is the radius of the largest disc $D(z_0, \delta(r))$ in which $|f(z)|>1$, and $\sigma(a, b)$ is non-Euclidean hyperbolic distance between a and b. We can map conformally this disc $D(z_0, \delta(r))$ onto a disc $|\zeta|<d(r)$ in ζ-plane by a transformation

$$\zeta=S(z)=(z-z_0)/(1-\bar{z}_0 z). \tag{4.3}$$

Then obviously $d(r)=\operatorname{th}\delta(r)$, where $\operatorname{th} x=(e^x-e^{-x})/(e^x+e^{-x})$. Further we define $F(\zeta)$ by $f(z)=F(\zeta), \zeta=S(z)$. Then $F(\zeta)$ is regular in $|\zeta|<d(r)$ and $|F(\zeta)|>1$ in $|\zeta|<d(r)$. Hence, by Lemma 1

$$d(r) \leqq \frac{2\,|F(0)|\log|F(0)|}{|F'(0)|}, \tag{4.4}$$

and for some ζ in $|\zeta|<d(r)$

$$\rho(F(\zeta)) \geqq \frac{\log|F(0)|}{10d(r)\log 2}. \tag{4.5}$$

Returning to z-plane, we get from (4.4) and (4.5)

$$d(r) \leqq \frac{2\,|f(z_0)|\log|f(z_0)|}{|f'(z_0)|(1-|z_0|^2)}, \tag{4.4$'$}$$

$$\frac{|1-\bar{z}_0 z|^2}{1-|z_0|^2}\rho(f(z)) \geqq \frac{\log|f(z_0)|}{10d(r)\log 2} \quad \text{for some } z \text{ in } D(z_0, \delta(z)). \tag{4.5$'$}$$

On the other hand, we have by (4.1) and (4.4)$'$

$$\text{th}\,\delta(r)=d(r) \leqq \frac{2\varphi(r)}{\varphi'(r)}\frac{1}{1-r^2}. \tag{4.6}$$

Hence, from (3.4)

$$\text{th}\,\delta(r)=d(r) \leqq \frac{2}{\alpha'}(1-r)\frac{1}{1-r^2} \leqq \frac{2}{\alpha'}<1. \tag{4.7}$$

Therefore, by (4.5)$'$

$$\rho(f(z)) \geqq \frac{\varphi(r)\alpha'}{20\log 2}\frac{1-r}{4}. \tag{4.8}$$

Using (3.5), we obtain

$$\rho(f(z)) \geqq \frac{\alpha'\beta}{80\log 2}\left(\frac{1}{1-r}\right)^{\lambda-1}. \tag{4.9}$$

Now setting $|z|=R$ for z satisfying (4.5)$'$, we get

$$r-d_2(r)<R<r+d_1(r)<1 \tag{4.10}$$

since $z \in D(z_0, \delta(r))$, where

$$d_1(r)=\frac{(1-|z_0|^2)\,\text{th}\,\delta(r)}{1+|z_0|\,\text{th}\,\delta(r)} \quad \text{and} \quad d_2(r)=\frac{(1-|z_0|^2)\,\text{th}\,\delta(r)}{1-|z_0|\,\text{th}\,\delta(r)}.$$

Then we note by (4.7) that $d_2(r)\to 0$ as $r\to 1$. Hence by (4.10) we see that $R\to 1$ as $r\to 1$. Here we consider two cases: 1) $r\geqq R$, 2) $r<R$.

Case 1). In this case, we get from (4.9)

$$\mu(R, f) \geqq \rho(f(z)) \geqq \frac{\alpha'\beta}{80\log 2}\left(\frac{1}{1-R}\right)^{\lambda-1} \tag{4.11}$$

since $1/(1-r)\geqq 1/(1-R)$.

Case 2). In this case, by (4.10)

$$1/(1-R)<1/(1-r-d_1(r)). \tag{4.12}$$

On the other hand, we have by (4.7) and the definition of $d_1(r)$

$$1-r-d_1(r)=1-r-\frac{(1-r^2)\,\text{th}\,\delta(r)}{1+r\,\text{th}\,\delta(r)} \geqq (1-r)\frac{\alpha'-2}{\alpha'+2}. \tag{4.13}$$

From (4.12) and (4.13), we get

$$\frac{1}{1-r} \geqq \frac{\alpha'-2}{\alpha'+2}\frac{1}{1-R}. \tag{4.14}$$

Thus by (4.9) and (4.14) we can obtain

$$\mu(R, f) \geqq \rho(f(z)) \geqq \frac{\alpha'\beta}{80 \log 2}\left(\frac{\alpha'-2}{\alpha'+2}\right)^{\lambda-1}\left(\frac{1}{1-R}\right)^{\lambda-1}. \tag{4.15}$$

In either case, therefore, we obtain from (4.11) and (4.15)

$$\varlimsup_{R\to 1} (1-R)^{\lambda-1}\mu(R, f) \geqq \frac{\alpha'\beta}{80 \log 2}\left(\frac{\alpha'-2}{\alpha'+2}\right)^{\lambda-1}. \tag{4.16}$$

Here α' can be taken as near to λ as we please. This proves our Theorem 1.

5. Corollaries of Theorem 1. Suppose that for functions meromorphic in $|z|<1$

$$\mu(r, f) = K(1-r)^{-p}, \tag{5.1}$$

where K is a positive constant and $1<p<\infty$. Then,

$$T(r, f) = O\{(1-r)^{-2p+2}\} \tag{5.2}$$

holds. Particularly, *if $f(z)$ is a meromorphic function of order λ $(p<\lambda\leqq\infty, p>0)$, from* (5.1) *and* (5.2)

$$\varlimsup_{r\to 1} (1-r)^{\frac{p}{2}+1}\mu(r, f) = \infty. \tag{5.3}$$

For this, we can get the following result by the same method as in Theorem 1.

Corollary 1. *If $f(z)$ is a regular function in $|z|<1$ and satisfies the condition* (5.1), *then*

$$T(r, f) = O\{(1-r)^{-p-1}\} \qquad (r\to 1). \tag{5.4}$$

This is a sharper estimate than (5.2) when $p\geqq 3$.

Proof. Suppose that for some positive constant β'

$$\varlimsup_{r\to 1} \frac{\log M(r, f)}{(1-r)^{-p-1}} > \beta' K.$$

Applying Lemma 2 with $\alpha=p+1, \alpha>\alpha'>2$, and $\beta=\beta' K$ to $\varphi(r)=\log M(r, f)$, (3.4) and (3.5) hold. Hence, by the same method that (4.11) and (4.15) were obtained, we can get

$$\mu(r, f) \geqq \frac{\alpha'\beta' K}{80 \log 2}\left(\frac{\alpha'-2}{\alpha'+2}\right)^{p}\left(\frac{1}{1-r}\right)^{p}. \tag{5.5}$$

Therefore, from our assumption we have

$$\beta' \leqq \frac{80 \log 2}{p+1}\left(\frac{p+3}{p-1}\right)^{p} < \frac{80 \log 3}{p+1}\left(\frac{p+3}{p-1}\right)^{p} \equiv \beta_0.$$

Hence we get for $\beta'=\beta_0$

$$\frac{\log M(r, f)}{(1-r)^{-p-1}} \leqq \beta' K. \tag{5.6}$$

Consequently, by a well-known inequality ([6], p. 220):

$$T(r, f) \leqq \log M(r, f) \leqq \frac{R+r}{R-r} T(R, f) \qquad (r<R) \tag{5.7}$$

we obtain (5.4). This completes the proof.

Further, we can get easily the next relation from Theorem 1.

Corollary 2. *Suppose that $f(z)$ is a regular function of order $\lambda=\infty$ in $|z|<1$. Then, for arbitrarily large number $N>0$*

$$\overline{\lim_{r\to 1}}\,(1-r)^N\mu(r, f)=\infty. \tag{5.8}$$

6. Further Results. Next we shall show the following inequality which holds for regular functions of finite order.

Theorem 2. *Let $f(z)$ be a regular function of order λ $(0\leqq\lambda<\infty)$ in $|z|<1$. Then, for any positive number ε,*

$$\overline{\lim_{r\to 1}}\frac{(1-r)\rho(f(z))}{\exp\left[C(1-r)^{-\lambda-1-\varepsilon}\right]}=O(1) \tag{6.1}$$

holds, where C is a positive constant depending on $f(z)$ and ε.

Proof. By Cauchy's integral formula, we write

$$f'(z)=\frac{1}{2\pi i}\int_{|\zeta-z|=r'-r}\frac{f(\zeta)}{(\zeta-z)^2}d\zeta, \tag{6.2}$$

where $r'=(1+r)/2$ and $r=|z|$. Hence we get

$$|f'(z)|\leqq\frac{1}{2\pi}\int_{|\zeta-z|=r'-r}\frac{|f(\zeta)|}{|\zeta-z|^2}|d\zeta|$$

$$\leqq\frac{M(r', f)}{2\pi(r'-r)^2}2\pi(r'-r)=\frac{2M(r', f)}{1-r}, \tag{6.3}$$

where $M(r', f)=\max\limits_{|z|=r'}|f(z)|$. On the other hand, by (5.7)

$$\log M(r', f)\leqq\frac{r''+r'}{r''-r'}T(r'', f), \tag{6.4}$$

where $r''=(1+r')/2$ and $r'=(1+r)/2$. Therefore we get

$$\log M(r', f)\leqq\frac{r''+r'}{r''-r'}T(r'', f)\leqq\frac{8}{1-r}T(r'', f). \tag{6.5}$$

Since $f(z)$ is of order λ, for any positive number ε there exists a value $r(\varepsilon)$ such that for all $r>r(\varepsilon)$

$$T(r, f)<(1-r)^{-\lambda-\varepsilon}. \tag{6.6}$$

Therefore using (6.5) and (6.6), we have

$$M(r', f)\leqq\exp\left[8\cdot 4^{\lambda+\varepsilon}(1-r)^{-\lambda-1-\varepsilon}\right]\qquad (r>r(\varepsilon)). \tag{6.7}$$

From (6.3) and (6.7), we obtain

$$\rho(f(z))\leqq|f'(z)|\leqq\frac{2}{1-r}\exp\left[C(1-r)^{-\lambda-1-\varepsilon}\right], \tag{6.8}$$

where $C=8\cdot 4^{\lambda+\varepsilon}$. Consequently we have (6.1).

From our proof of Theorem 2, we get:

Corollary 3. *If $f(z)$ is regular and of bounded characteristic in $|z|<1$, then*

$$\overline{\lim_{r\to 1}}\frac{(1-r)\rho(f(z))}{\exp\left[C(1-r)^{-1}\right]}=O(1). \tag{6.9}$$

7. W. K. Hayman recently proved the following ([7]).

Theorem D. *Suppose that $f(z)=\sum_{\nu=0}^{\infty}a_{n_\nu}z^{n_\nu}$ is mean p-valent in $|z|<1$* ([8], p. 23) *and that*

$$n_{\nu+1}-n_{\nu}\geqq C, \qquad (\nu\geqq\nu_0) \tag{7.1}$$

holds. Then

$$M(r,f)<A(p,C,\nu_0)\mu_p(1-r)^{-2p/C}, \qquad 0<r<1. \tag{7.2}$$

Here $\mu_p=\max_{0\leqq n\leqq p}|a_n|$, $M(r,f)=\max_{|z|=r}|f(z)|$ *and* $A(p,C,\nu_0)$ *denotes a particular constant depending on* p, C, ν_0 *only.*

From this Theorem D and our proof of Theorem 2 we obtain the following corollary.

Corollary 4. *Suppose that* $f(z)=\sum_{k=0}^{\infty}a_{n_k}z^{n_k}$ *is mean p-valent in* $|z|<1$ *and that*

$$n_{k+1}-n_k\geqq q \tag{7.3}$$

holds. Then we get

$$\overline{\lim_{r\to 1}}\,(1-r)^{\frac{2p}{q}+1}\rho(f(z))=O(1), \tag{7.4}$$

where $0<p<\infty$ *and* q *is an integer such that* $q\geqq 1$.

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