181. *On the Analyticity and the Unique Continuation Theorem for Solutions of the Navier-Stokes Equation*

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(Comm. by Zyoiti SUETUNA, M.J.A., Nov. 13, 1967)

1. Introduction. Consider the Navier-Stokes equation

(1) $\boldsymbol{u}_t+(\boldsymbol{u}\cdot\mathrm{grad})\boldsymbol{u}=\Delta\boldsymbol{u}-\nabla p+\boldsymbol{f},\ \mathrm{div}\,\boldsymbol{u}=0,\ x\in G,\ 0<t<T,$

and the condition of adherence at the boundary

(2) $\boldsymbol{u}=0$ on the boundary of G.

Here G is a connected component of exteriors (or interiors) of a bounded hypersurface of class C^2, $\boldsymbol{u}$ and $\boldsymbol{f}$ are 3-dimensional real vector functions of x and t, and p is a scalar function of x and t. We are mainly concerned with the question whether a nonconstant flow of incompressible fluid, subject to the Navier-Stokes equation (1) with $f=0$ and the condition (2) of adherence at the boundary, can ever come to rest in a finite time on some potion of G. Before stating our results, we shall define function spaces, and fix our notations. For any open set Q in R^n, $\boldsymbol{W}^{k,p}(Q)$ ($k\geqslant 0, 1\leqslant p<\infty$) is the set of all complex-valued vector functions in $\boldsymbol{L}^p(Q)$ for which distribution derivatives of up to order k lie in $\boldsymbol{L}^p(Q)$. $\boldsymbol{W}^{k,p}(Q)$ ($k>0$) is the set of all distributions $\boldsymbol{u}$ such that $|\langle\boldsymbol{u},\boldsymbol{\varphi}\rangle$[1]$|\leqslant C\,\|\boldsymbol{\varphi}\|_{L^p}$ for $\boldsymbol{\varphi}$ in $C_0^\infty(Q)$, C being a positive constant, where $\|\boldsymbol{\varphi}\|_{L^p}$ is the L^p-norm of $\boldsymbol{\varphi}$. $\boldsymbol{W}^{k,p}_{\mathrm{loc}}(Q)$ ($k=0,\pm 1,\cdots$) is the set of all distribution $\boldsymbol{u}$ on Q which coincide on some neighborhood of each point of Q with elements of $\boldsymbol{W}^{k,p}(Q)$. The set of all 3-dimensional real vector functions φ such that $\varphi\in C_0^\infty(G)$, and $\mathrm{div}\,\varphi=0$, is denoted by $C_{0,s}^\infty(G)$. Let $\boldsymbol{L}_s^2=\boldsymbol{L}_s^2(G)$ be the closure of $C_{0\ s}^\infty(G)$ in $\boldsymbol{L}^2(G)$. Let P be the orthogonal projection from $\boldsymbol{L}^2(G)$ onto $\boldsymbol{L}_s^2$. By A we denote the Friedrichs extension of the symmetric operator $-P\Delta$ in $\boldsymbol{L}_s^2$ defined for every $\boldsymbol{u}$ such that $\boldsymbol{u}\in C^2(G)\cap C^1(G^a)$, $\mathrm{div}\,\boldsymbol{u}=0$, and $\boldsymbol{u}=0$ on the boundary of G, G^a being the closure of G. By X_γ we denote the set of all $\boldsymbol{u}$ in $D(A^\gamma)$ with the norm $\|\boldsymbol{u}\|_{X_\gamma}=\|A^\gamma\boldsymbol{u}\|+\|\boldsymbol{u}\|$, $D(A^\gamma)$ being the domain of A^γ, where γ is any number with $3/4<\gamma<1$. We let $\boldsymbol{X}=\boldsymbol{X}_{4/5}$. Here $\|\cdot\|$ is the norm of the Hilbert space $\boldsymbol{L}^2(G)$ with the scalar product $(\cdot,\cdot)$. Let $\boldsymbol{H}_{0,s}^1=\boldsymbol{H}_{0,s}^1(G)$ be the completion of the set $C_{0,s}^1(G)$ of all solenoidal ($\mathrm{div}\,\boldsymbol{u}=0$) functions in C_0^1 with the norm $\|\nabla\boldsymbol{u}\|+\|\boldsymbol{u}\|$. Now our results are as follows.

1) $\langle\boldsymbol{u},\boldsymbol{\varphi}\rangle$ denotes the value of the functional $\boldsymbol{u}$ at $\boldsymbol{\varphi}$.

Theorem 1. *Let there exists an analytic extension $\boldsymbol{f}(z)=\boldsymbol{f}(\cdot, z)$ of $\boldsymbol{f}(t)=\boldsymbol{f}(\cdot, t)$ such that $\boldsymbol{f}(z)$ is an $\boldsymbol{L}^2_s$-valued holomorphic function of z in some neighbourhood Ω of $(0, T)$. Let $\boldsymbol{u}$ be a solution of* (1), (2) *such that $\boldsymbol{u}(\cdot, t)$ is an $\boldsymbol{H}^1_{0,s}$-valued continuous function of t in $(0, T)$. If $\boldsymbol{f}(x, z)$ is analytic in (x, z) in some nonempty open subset $G_0\times\Omega$ of $G\times\Omega$, then there exists an analytic (in x and t) function $\boldsymbol{u}^*(x, t)$ on $G_0\times(0, T)$ such that for each t in $(0, T)$ $\boldsymbol{u}^*(x, t)=\boldsymbol{u}(x, t)$, $x\in G_0$, after a correction on a null set of the space R^3.*

Theorem 2. *Let $\boldsymbol{u}$ be a solution of* (1), (2) *with $\boldsymbol{f}=0$ such that $\boldsymbol{u}(\cdot, t)$ is an $\boldsymbol{H}^1_{0,s}$-valued continuous function of t in $(0, T)$. If there exist a nonempty open set G_1 in G and a t_1 with $0<t_1<T$ such that $u(x, t_1)=0$, $x\in G_1$, then $\boldsymbol{u}$ vanishes identically in $G\times(0, T)$.*

Here by a solution $\boldsymbol{u}$ of (1), (2), we mean a locally square summable function $\boldsymbol{u}(x, t)$ on $G\times(0, T)$ with the following properties: (i) $\boldsymbol{u}(x, t)$ is weakly divergent free, i.e. $\int(\boldsymbol{u}, \operatorname{grad}\omega)dt=0$ for all scalar function $\omega\in C_0^\infty(G\times(0,T))$, (ii) $\int\{(\boldsymbol{u}, \boldsymbol{\Phi}_t)+(\boldsymbol{u}, \Delta\boldsymbol{\Phi})+(\boldsymbol{u}, \boldsymbol{u}\cdot\operatorname{grad}\boldsymbol{\Phi})+(\boldsymbol{f},\boldsymbol{\Phi})\}dt=0$ for all C^∞ vectors $\boldsymbol{\Phi}$ which are solenoidal and have compact support in $G\times(0, T)$.

It is to be noted that the above theorems give partial answers to Serrin's conjectures [1].

2. Lemmas for the proof of the theorems. **Lemma 1.** (a) *$D(A^{1/2})=H^1_{0,s}$, and $\|A^{1/2}\boldsymbol{u}\|=\|\nabla\boldsymbol{u}\|$ for $\boldsymbol{u}\in D(A^{1/2})$.* (b) *For any bounded open set E in G, its closure being contained in G, there exists a constant $C=C(E)$ such that* $\operatorname{ess.sup}_{x\in E}|v(x)|\le C\|v\|_X$, $v\in X$.

For the proof see Fujita-Kato [2].

Lemma 2. *There exists an analytic extension $\boldsymbol{u}(z)=\boldsymbol{u}(\cdot, z)$ of $\boldsymbol{u}(t)=\boldsymbol{u}(\cdot, t)$ $(t\in(0, T))$ such that $\boldsymbol{u}(z)$ is an $\boldsymbol{X}$-valued holomorphic function of z in some neighbourhood U, contained in Ω, of $(0, T)$ in the complex plane, satisfying the equation $\partial(\boldsymbol{u}, \boldsymbol{\varphi})/\partial z=-(\boldsymbol{u}, A\boldsymbol{\varphi})-((\boldsymbol{u}\cdot\operatorname{grad})\boldsymbol{u}, \boldsymbol{\varphi})+(\boldsymbol{f}, \boldsymbol{\varphi})$ for $\boldsymbol{\varphi}$ in $C^\infty_{0,s}(G)$ and z in U.*

An outline of the proof of Lemma 2 will be given in section 4.

3. Proof of Theorem 1. We set $\boldsymbol{v}(x, z)=\operatorname{rot}_x\boldsymbol{u}(x, z)(\equiv\operatorname{rot}\boldsymbol{u}(x, z))$, $\boldsymbol{u}(x, \xi, \eta)=\boldsymbol{u}(x, \xi+i\eta)$, $\boldsymbol{v}(x, \xi, \eta)=\boldsymbol{v}(x, \xi+i\eta)$, and $\boldsymbol{f}(x, \xi, \eta)=\boldsymbol{f}(x, \xi+i\eta)$, $x\in G$, $\xi+i\eta\in U$. Then for any $\boldsymbol{\varphi}$ in $C_0^\infty(G)(\boldsymbol{u}(\cdot, \xi, \eta), \boldsymbol{\varphi})$, and $(\boldsymbol{v}(\cdot, \xi, \eta), \boldsymbol{\varphi})$ are harmonic functions of ξ and η, since $\boldsymbol{u}(\cdot, z)$, and $\boldsymbol{v}(\cdot, z)$ are $\boldsymbol{L}^2_s$-valued holomorphic functions of z. Hence we have

$$((\boldsymbol{u}, [\partial^2/\partial\xi^2+\partial^2/\partial\eta^2]\boldsymbol{\varphi}\psi))=0, \tag{3}$$

$$((\boldsymbol{v}, [\partial^2/\partial\xi^2+\partial^2/\partial\eta^2]\boldsymbol{\varphi}\psi))=0 \tag{4}$$

for any vector $\boldsymbol{\varphi}$ in $C_0^\infty(G_0)$ and any scalar ψ in $C_0^\infty(U_0)$, where $U_0=\{(\xi, \eta); \xi+i\eta\in U\}$ and $((\cdot, \cdot))$ is the scalar product in $\boldsymbol{L}^2(G_0\times U_0)$. Using the relation $\operatorname{rot}\operatorname{rot}=\operatorname{grad}\operatorname{div}-\Delta$, we have $(\boldsymbol{u}, -\Delta\boldsymbol{\varphi})=(\boldsymbol{v}, \operatorname{rot}\boldsymbol{\varphi})$, $\boldsymbol{\varphi}\in C_0^\infty(G_0)$, since $(\boldsymbol{u}, \operatorname{grad}\operatorname{div}\boldsymbol{\varphi})=0$ in virtue of the fact that $\boldsymbol{u}\in\boldsymbol{L}^2_s$.

Consequently,

$$(5)\qquad ((\boldsymbol{u}, \Delta\boldsymbol{\varphi}\psi)) = -((\boldsymbol{v}, \operatorname{rot}\boldsymbol{\varphi}\psi))$$

for $\boldsymbol{\varphi}$ in $C_0^\infty(G_0)$ and ψ in $C_0^\infty(U_0)$. On the other hand, noting that $\operatorname{rot}\boldsymbol{\varphi} \in \boldsymbol{L}_s^2$ for $\boldsymbol{\varphi} \in C_0^\infty(G_0)$, we have, by Lemma 2, $\partial(\boldsymbol{u}, \operatorname{rot}\boldsymbol{\varphi})/\partial\xi = (\boldsymbol{u}, \Delta \operatorname{rot}\boldsymbol{\varphi}) - ((\boldsymbol{u}\cdot\operatorname{grad})\boldsymbol{u}, \operatorname{rot}\boldsymbol{\varphi}) + (\boldsymbol{f}, \operatorname{rot}\boldsymbol{\varphi})$, $(\xi, \eta) \in U_0$, so that

$$(6)\qquad ((\boldsymbol{v}, [\partial/\partial\xi + \Delta]\boldsymbol{\varphi}\psi)) - (((\boldsymbol{u}\cdot\operatorname{grad})\boldsymbol{u}, \operatorname{rot}\boldsymbol{\varphi}\psi)) + ((\operatorname{rot}\boldsymbol{f}, \boldsymbol{\varphi}\psi)) = 0$$

for any $\boldsymbol{\varphi}$ in $C_0^\infty(G_0)$ and any ψ in $C_0^\infty(U_0)$. By adding (3) to (5), and (4) to (6), we have

$$((\boldsymbol{u}, [\partial^2/\partial\xi^2 + \partial^2/\partial\eta^2 + \Delta]\boldsymbol{\varphi}\psi)) + ((\boldsymbol{v}, \operatorname{rot}\boldsymbol{\varphi}\psi)) = 0$$

and

$$((\boldsymbol{v}, [\partial^2/\partial\xi^2 + \partial^2/\partial\eta^2 + \Delta + \partial/\partial\xi]\boldsymbol{\varphi}\psi)) - (((\boldsymbol{u}\cdot\operatorname{grad})\boldsymbol{u}, \operatorname{rot}\boldsymbol{\varphi}\psi)) + ((\operatorname{rot}\boldsymbol{f}, \boldsymbol{\varphi}\psi)) = 0,$$

for $\boldsymbol{\varphi} \in C_0^\infty(G_0)$, and $\psi \in C_0^\infty(U_0)$. Since the totality of finite sums $\sum \boldsymbol{\varphi}_j\psi_j$ with $\boldsymbol{\varphi}_j \in C_0^\infty(G_0)$ and $\psi_j \in C_0^\infty(U_0)$ is dense in $C_0^\infty(G_0\times U_0)$ in the topology of $D(G_0\times U_0)$ (see L. Schwartz [3] p. 107), we obtain

$$(7)\qquad ((\boldsymbol{u}, [\partial^2/\partial\eta^2 + \partial^2/\partial\xi^2 + \Delta]\Phi)) + ((\boldsymbol{v}, \operatorname{rot}\Phi)) = 0,$$

$$(8)\qquad ((\boldsymbol{v}, [\partial^2/\partial\xi^2 + \partial^2/\partial\eta^2 + \Delta + \partial/\partial\xi]\Phi)) - ((([\boldsymbol{u}\cdot\operatorname{grad}]\boldsymbol{u}, \operatorname{rot}\Phi)) + ((\operatorname{rot}\boldsymbol{f}, \Phi)) = 0$$

for any Φ in $C_0^\infty(G_0\times U_0)$. Let $K = E\times F$ be any bounded open set, its closure being contained in $G_0\times U_0$. Then we shall show that

$$(9)\qquad \boldsymbol{u} \in \boldsymbol{W}^{2,10/3}(K),\ \boldsymbol{v} \in \boldsymbol{W}^{1,10/3}(K).$$

Since $\boldsymbol{u}(\cdot, z)$ is an X-valued, and so $H_{0,s}^1$-valued, holomorphic function of z, we see that $\boldsymbol{v}(\cdot, z)$ is an $\boldsymbol{L}^2(E)$-valued holomorphic function of z in view of $\operatorname{rot}\boldsymbol{u} = \boldsymbol{v}$, and that $\boldsymbol{u}(\cdot, z)$ is an $\boldsymbol{L}^\infty(E)$-valued continuous function of z in view of the fact that $\operatorname{ess.sup}_{x\in E} |\boldsymbol{u}(x, z)| \leqslant C\,\|\boldsymbol{u}(\cdot, z)\|_X$ for some constant C, independent of z, by Lemma 1. Hence $\boldsymbol{v} \in \boldsymbol{L}^2(K)$ and $\boldsymbol{u} \in \boldsymbol{L}^\infty(K)$. Since

$$\|(\boldsymbol{u}\cdot\operatorname{grad})\boldsymbol{u}\|_{L^2(K)} \leqslant (\operatorname{ess.sup}_K |\boldsymbol{u}|)\,\|\nabla\boldsymbol{u}\| \leqslant (\operatorname{ess.sup} |\boldsymbol{u}|)\,\|\boldsymbol{u}\|_X$$

by Lemma 1, we have $(\boldsymbol{u}\cdot\operatorname{grad})\boldsymbol{u} \in \boldsymbol{L}^2(K)$, from which it follows that $\operatorname{rot}\boldsymbol{f} - \operatorname{rot}(\boldsymbol{u}\cdot\operatorname{grad})\boldsymbol{u} \in \boldsymbol{W}^{-1,2}(K)$. We have, by (8), $(\partial^2/\partial\xi^2 + \partial^2/\partial\eta^2 + \Delta - \partial/\partial\xi)\,\boldsymbol{v} \in \boldsymbol{W}_{\mathrm{loc}}^{-1,2}(K)$. Hence, applying the interior regularity theorem (I. R. THM.) of weak solutions of elliptic equations, we have $\boldsymbol{v} \in \boldsymbol{W}_{\mathrm{loc}}^{1,2}(K)$, so that $(\partial^2/\partial\xi^2 + \partial^2/\partial\eta^2 + \Delta - \partial/\partial\xi)\boldsymbol{u} = \operatorname{rot}\boldsymbol{v} \in \boldsymbol{L}_{\mathrm{loc}}^2(K)$ by (8). Hence $\boldsymbol{u} \in \boldsymbol{W}_{\mathrm{loc}}^{2,2}(K)$. By Sobolev's lemma, $\boldsymbol{v} \in \boldsymbol{W}_{\mathrm{loc}}^{1,2}(K)$ implies $\boldsymbol{v} \in \boldsymbol{L}_{\mathrm{loc}}^{10/3}(K)$. Also we have $\boldsymbol{u} \in \boldsymbol{W}_{\mathrm{loc}}^{1,10/3}(K)$. By the arbitrariness of the choice of K, $\boldsymbol{v} \in \boldsymbol{L}^{10/3}(K)$ and $\boldsymbol{u} \in \boldsymbol{W}^{1,10/3}(K)$. Since

$$\|(\boldsymbol{u}\cdot\operatorname{grad})\boldsymbol{u}\|_{L^{10/3}(K)} \leqslant (\operatorname{ess.sup}_K |\boldsymbol{u}|)\times\|\boldsymbol{u}\|_{W^{1,10/3}(K)},$$

we have $(\boldsymbol{u}\cdot\operatorname{grad})\boldsymbol{u} \in \boldsymbol{L}^{10/3}(K)$, and so $\operatorname{rot}\boldsymbol{f} - \operatorname{rot}(\boldsymbol{u}\cdot\operatorname{grad})\boldsymbol{u} \in \boldsymbol{W}_{\mathrm{loc}}^{-1,10/3}(K)$. Hence applying the I. R. THM. to Eq. (7), and to Eq. (8) once more, we have $\boldsymbol{v} \in \boldsymbol{W}_{\mathrm{loc}}^{1,10/3}(K)$ and $\boldsymbol{u} \in \boldsymbol{W}_{\mathrm{loc}}^{2,10/3}(K)$. By the arbitrariness of the choice of K, we have (9). Next we shall show that if $\boldsymbol{u} \in \boldsymbol{W}^{k+1,10/3}(K)$ and $\boldsymbol{v} \in \boldsymbol{W}^{k,10/3}(K)$, then $\boldsymbol{u} \in \boldsymbol{W}^{k+2,10/3}(K)$ and $\boldsymbol{v} \in \boldsymbol{W}^{k+1,10/3}(K)$, k being a positive integer. Let $D^{\alpha}=(\partial/\partial x_1)^{\alpha_1}(\partial/\partial x_2)^{\alpha_2}(\partial/\partial x_3)^{\alpha_3}$, $\alpha=(\alpha_1, \alpha_2, \alpha_3)$. Then we have $D^{\alpha}[(\boldsymbol{u}\cdot\mathrm{grad})\boldsymbol{u}]=(D^{\alpha}\boldsymbol{u}\cdot\mathrm{grad})\boldsymbol{u}+\sum_{\beta<\alpha}C_{\beta}(D^{\beta}\boldsymbol{u}\cdot\mathrm{grad})D^{\beta-\alpha}\boldsymbol{u}$ for $|\alpha|=\alpha_1+\alpha_2+\alpha_3\leqslant k$, C_{β} being a constant independent of $\boldsymbol{u}$. Since

$$\begin{aligned}\|(D^{\alpha}\boldsymbol{u}\cdot\mathrm{grad})\boldsymbol{u}\|_{L^{10/3}(K)}&\leqslant\|D^{\alpha}\boldsymbol{u}\|_{L^{20/3}(K)}\cdot\|\boldsymbol{u}\|_{W^{1,20/3}(K)}\\&\leqslant C\|\boldsymbol{u}\|_{W^{k+1,10/3}(K)}\|\boldsymbol{u}\|_{W^{k+1,10/3}(K)}\end{aligned}$$

by the Hölder inequality and Sobolev's lemma, we have $(D^{\alpha}\boldsymbol{u}\cdot\mathrm{grad})\boldsymbol{u}\in\boldsymbol{L}^{10/3}(K)$. On the other hand, since

$$\|(D^{\beta}\boldsymbol{u}\cdot\mathrm{grad})D^{\alpha-\beta}\boldsymbol{u}\|_{L^{10/3}(K)}\leqslant C(\mathrm{ess.\,sup}_K|D^{\beta}\boldsymbol{u}|)\|\boldsymbol{u}\|_{W^{k+1,10/3}(K)}$$

for $\beta<\alpha$, C being a constant independent of $\boldsymbol{u}$, we have $(D^{\beta}\boldsymbol{u}\cdot\mathrm{grad})D^{\alpha-\beta}\boldsymbol{u}\in\boldsymbol{L}^{10/3}(K)$. Here we used the fact that $\boldsymbol{u}\in\boldsymbol{W}^{k+1,10/3}(K)$ implies $D^{\beta}\boldsymbol{u}\in\boldsymbol{L}^{\infty}(K)$, $\beta<\alpha$, by Sobolev's lemma. Hence $(\boldsymbol{u}\cdot\mathrm{grad})\boldsymbol{u}\in\boldsymbol{W}^{k,10/3}(K)$, so that $\mathrm{rot}\,\boldsymbol{f}-\mathrm{rot}\,((\boldsymbol{u}\cdot\mathrm{grad})\boldsymbol{u})\in\boldsymbol{W}^{k-1,10/3}(K)$. Hence applying the I. R. THM. to Eq. (7), and to Eq. (8), we have $\boldsymbol{v}\in\boldsymbol{W}^{k+1,10/3}(K)$ and $\boldsymbol{u}\in\boldsymbol{W}^{k+2,10/3}(K)$. Hence $\boldsymbol{u}\in\boldsymbol{W}^{k+1,10/3}(K)$ and $\boldsymbol{v}\in\boldsymbol{W}^{k,10/3}(K)$ for arbitrary positive integer k, by (9). By Sobolev's lemma there exist $\boldsymbol{u}^*\in\boldsymbol{C}^{\infty}(K)$, $\boldsymbol{v}^*\in\boldsymbol{C}^{\infty}(K)$ such that $\boldsymbol{u}^*=\boldsymbol{u}$, and $\boldsymbol{v}^*=\boldsymbol{v}$ after a correction on a null set of the space R^5. Since $\mathrm{rot}\,((\boldsymbol{u}\cdot\mathrm{grad})\boldsymbol{u})=(\boldsymbol{u}\cdot\mathrm{grad})\,\mathrm{rot}\,\boldsymbol{u}-(\sum_{\alpha=1}^{3}(\mathrm{rot}\,\boldsymbol{u})_{\alpha}\cdot\partial\boldsymbol{u}_{\beta}/\partial x_{\alpha})$, we see, by (7) and (8), that a vector $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ satisfies a non-linear analytic elliptic system $[\partial^2/\partial\xi^2+\partial^2/\partial\eta^2+\Delta]\boldsymbol{u}^*+\mathrm{rot}\,\boldsymbol{v}^*=0$, $[\partial^2/\partial\xi^2+\partial^2/\partial\eta^2+\Delta+\partial/\partial\xi]\boldsymbol{v}^*-(\boldsymbol{u}^*\cdot\mathrm{grad})\boldsymbol{v}^*-(\sum_{\alpha=1}^{3}\boldsymbol{v}_{\alpha}^*\cdot\partial\boldsymbol{u}_{\beta}^*/\partial x_{\alpha})-\mathrm{rot}\,\boldsymbol{f}=0$ in $G_0\times U_0$. Applying the theorem on the analyticity of solutions of a non-linear analytic elliptic system (see Morrey [5]), we see that $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ is analytic in x, ξ, η in the interior of $G_0\times U_0$. Since $(\boldsymbol{u}(\cdot, z), \boldsymbol{\varphi})$ and $(\boldsymbol{v}(\cdot, z), \boldsymbol{\varphi})$ are analytic in z for $\boldsymbol{\varphi}$ in $C_0^{\infty}(G_0)$, we have $(\boldsymbol{u}(\cdot, \xi, 0), \boldsymbol{\varphi})=(\boldsymbol{u}^*(\cdot, \xi, 0), \boldsymbol{\varphi})$ and $(\boldsymbol{v}(\cdot, \xi, 0), \boldsymbol{\varphi})=(\boldsymbol{v}^*(\cdot, \xi, 0), \boldsymbol{\varphi})$. Hence for each t in $(0, T)$ $\boldsymbol{u}(x, t)=\boldsymbol{u}^*(x, t, 0)$ and $\boldsymbol{v}(x, t)=\boldsymbol{v}^*(x, t, 0)$, $x\in G_0$, after a correction on a null set of the space R^3. This shows that $\boldsymbol{u}\,(x, t)$ and $\boldsymbol{v}(x, t)$ are analytic in x and t, $x\in G_0$, $t\in(0, T)$. Theorem 1 is thus proved.

Proof of Theorem 2. Since $\boldsymbol{u}(x, t)$ is analytic in x and t $(x\in G, t\in(0, T))$ by Theorem 1, the assumption $\boldsymbol{u}(x, t_1)=0$, $x\in G_1$, implies that $\boldsymbol{u}(x, t_1)=0$, $x\in G$, so that $\boldsymbol{v}(x, t_1)=0$, $x\in G$. Since $\boldsymbol{v}$ satisfies the equation $\partial\boldsymbol{v}/\partial t=\Delta\boldsymbol{v}-\mathrm{rot}\,((\boldsymbol{u}\cdot\mathrm{grad})\boldsymbol{u})$, we have $\boldsymbol{v}_t(x, t_1)=0$, and so $\mathrm{rot}\,\boldsymbol{u}_t(x, t_1)=0$, $x\in G$. Since $\boldsymbol{u}_t(x, t)\in H_{0,s}^1(G)$ and $\mathrm{div}\,\boldsymbol{u}_t(x, t)=0$ by the $\boldsymbol{H}_{0,s}^1(G)$-valued analyticity of $\boldsymbol{u}(\cdot, t)$ (see Lemma 2), we have $\boldsymbol{u}_t(x, t_1)=0$, $x\in G$. Hence $\boldsymbol{v}_{tt}(x, t_1)=\Delta\boldsymbol{v}_t(x, t_1)-\mathrm{rot}\,((\boldsymbol{u}_t\cdot\mathrm{grad})\boldsymbol{u})\cdot(x, t_1)-\mathrm{rot}\,((\boldsymbol{u}\cdot\mathrm{grad})\boldsymbol{u}_t)\cdot(x, t_1)=0$, $x\in G$. Taking into account that $\boldsymbol{u}_{tt}(x, t_1)\in\boldsymbol{H}_{0,s}^1(G)$ and $\mathrm{div}\,\boldsymbol{u}_{tt}(x, t_1)=0$, we have $\boldsymbol{u}_{tt}(x, t_1)=0$, $x\in G$. Applying the same argument, we have $(\partial/\partial t)^k\boldsymbol{u}(x, t_1)=0$, $x\in G$, $k=1, 2, \cdots$. For any $\boldsymbol{\varphi}$ in $C_0^{\infty}(G)$ $(\boldsymbol{u}(\cdot, t), \boldsymbol{\varphi})$ has a zero of infinite order at t_1. By the analyticity in t of $(\boldsymbol{u}(\cdot, t), \boldsymbol{\varphi})$, $(\boldsymbol{u}(\cdot, t), \boldsymbol{\varphi})=0$ for any $\boldsymbol{\varphi}$ in $C_0^{\infty}(G)$ and any t in $(0, T)$, showing that $\boldsymbol{u}(x, t)$ vanishes identically on

$G\times(0, T)$. Theorem 2 is thus proved.

4. Proof of Lemma 2.[2] In this section we shall outline the proof of Lemma 2. At first we note that $\boldsymbol{u}(t)$ is an $\boldsymbol{X}$-valued continuous function of t in $(0, T)$, satisfying the equation

$$\boldsymbol{u}(t)=\exp(-tA)\boldsymbol{u}(T_0)+\int_{T_0}^{t}\exp(-(t-s))\{f(s)+F[\boldsymbol{u}(s)]\}ds,$$

$T_0\leqslant t\leqslant T$, where T_0 is any number in $(0, T)$, and $F[\boldsymbol{v}]=P((\boldsymbol{v}\cdot\mathrm{grad})\boldsymbol{v})$; see Fujita-Kato [2]. Let ε be an arbitrary number with $0<\varepsilon<T/2$, and θ be a number such that the set $\{z;\ \varepsilon\leqslant\mathrm{Re}\,z\leqslant T-\varepsilon,\ |z-\varepsilon|\cos\theta\leqslant\mathrm{Re}\,z-\varepsilon\}$ is contained in Ω. We set $S(\varepsilon, \delta;\ T_0)=\{z;\ T_0\leqslant\mathrm{Re}\,z\leqslant T_0+\delta,\ |z-T_0|\cos\theta\leqslant\mathrm{Re}\,z-T_0\}$. Let $\{\boldsymbol{u}_{N,k}(z;\ T_0);\ N=1, 2, \cdots, k=1, 2, \cdots\}$ be a sequence of $\boldsymbol{X}$-valued functions defined through $\boldsymbol{u}_{N,0}(z;\ T_0)=0$ and

$$\boldsymbol{u}_{N,k}(z;\ T_0)=\exp(-(z-T_0)A_N)\boldsymbol{u}(T_0)$$
$$+\int_\gamma\exp(-(z-\zeta)A_N)\{\boldsymbol{f}(\zeta)+F[\boldsymbol{u}_{N,k-1}(\zeta)]\}d\zeta,\qquad k\geqslant 1,\ z\in S(\varepsilon, \delta;\ T_0),$$

the path γ of integration being the segment $[T_0, z]$, where $A_N=\int_0^N\lambda\, dE(\lambda)$, $E(\lambda)$ being the spectral family associated with A. Then there exists a $\delta=\delta(\varepsilon)>0$, independent of N, such that for any T_0 with $\varepsilon\leqslant T_0\leqslant T$ $\boldsymbol{u}_{N,k}(z;\ T_0)$ are $\boldsymbol{X}$-valued holomorphic functions of z, converging uniformly on $S(\varepsilon, \delta;\ T_0)$ to a limit $\boldsymbol{u}_N(z;\ T_0)$ as $k\longrightarrow\infty$ in the norm of $\boldsymbol{X}$. Hence $\boldsymbol{u}_N(z;\ T_0)$ are $\boldsymbol{X}$-valued holomorphic (continuous) functions of z in the interior of $S(\varepsilon, \delta;\ T_0)$ (on $S(\varepsilon, \delta;\ T_0)$), satisfying the equation

$$\boldsymbol{u}_N(z;\ T_0)=\exp(-zA_N)\boldsymbol{u}(T_0)+\int_\gamma\exp(-(z-\zeta)A_N)\{\boldsymbol{f}(\zeta)+F[\boldsymbol{u}_N(\zeta;T_0)]\}d\zeta.$$

It is easy to see that $\boldsymbol{u}_N(z;\ T_0)$ converges uniformly on $S(\varepsilon, \delta;\ T_0)$ to a limit $\boldsymbol{u}_\infty(z;\ T_0)$ as $N\longrightarrow\infty$ in the norm of $\boldsymbol{X}$. This limit $\boldsymbol{u}_\infty(z;\ T_0)$ satisfies the equation

$$\boldsymbol{u}_\infty(z;\ T_0)=\exp(-zA)\boldsymbol{u}(T_0)+\int_\gamma\exp(-(z-\zeta)A)\{\boldsymbol{f}(\zeta)+F[\boldsymbol{u}_\infty(\zeta;\ T_0)]\}d\zeta.$$

In particular $\boldsymbol{u}_\infty(t;\ T_0)$ satisfies the equation

$$(10)\quad \boldsymbol{u}_\infty(t;\ T_0)=\exp(-tA)\boldsymbol{u}(T_0)+\int_{T_0}^{t}\exp(-(t-s)A)\{\boldsymbol{f}(s)+F[\boldsymbol{u}_\infty(s)]\}ds$$

for t in $[T_0, T_0+\delta)$. It is known that Eq. (10) has a unique solution within the class $C([T_0, T_0+\delta);\boldsymbol{X})$, and that $\boldsymbol{u}(t)$ is an $\boldsymbol{X}$-valued continuous function of t in $[T_0, T_0+\delta)$, satisfying Eq. (10). Hence $\boldsymbol{u}_\infty(t;\ T_0)=\boldsymbol{u}(t)$ for t in $[T_0, T_0+\delta)$, so that $\boldsymbol{u}_\infty(z;\ T_0)=\boldsymbol{u}_\infty(z;\ T_1)$ for $z\in S(\varepsilon, \delta;\ T_0)\cap S(\varepsilon', \delta';\ T_1)$. We define $U=\cup S(\varepsilon, \delta;\ T_0)$ $(0<\varepsilon<T/2,\ 0<T_0<T)$ and $\hat{\boldsymbol{u}}(z)=\boldsymbol{u}_\infty(z;\ T_0)$ for z in $S(\varepsilon, \delta;\ T_0)$. Then $\hat{\boldsymbol{u}}(z)$ is an $\boldsymbol{X}$-valued holomorphic function, defined in U, with desired properties.

Acknowledgements. The author wishes to express his hearty thanks to Prof. H. Fujita who showed an interest in this work

[2] Details will be published elesewhere.

and stimulated him, and also to Prof. S. Itô for kind advices on the proof of Lemma 2.

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