211. A Product Theorem Concerning Some Generalized Compactness Properties[1)]

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1. **Introduction.** In the last forty years a number of product theorems concerning compact topological spaces have been proven. In particular, Tychonoff [9] showed that the product of two compact spaces is a compact space, Dieudonné [1] showed that the product of a compact space and a paracompact space is a paracompact space, and Dowker [2] showed that the product of a compact space and a countably paracompact space is a countably paracompact space. These are three of the best-known theorems of the type: If X is a compact topological space and Y is a topological space with some generalized compactness property π, then the product space $X\times Y$ has the property π. The purpose of this paper is to prove a general theorem of this type and also to offer a unified approach to many generalized compactness properties.

2. **A characterization of some generalized compactness properties.** For each topological space X, let $\mathfrak{P}(X)$ be the set of all subsets of X. Let $\mathfrak{T}$ be the class of all topological spaces, let $\mathfrak{S}=\cup\{\mathfrak{P}\mathfrak{P}\mathfrak{P}(X):X\in\mathfrak{T}\}$ and let $Q:\mathfrak{T}\to\mathfrak{S}$ be a function with $Q(X)\in\mathfrak{P}\mathfrak{P}\mathfrak{P}(X)$ whenever $X\in\mathfrak{T}$.

Definition 1. *Q is* slattable *over X if and only if, whenever Y is a topological space and $\Lambda\in Q(X)$, there exists $\Gamma\in Q(X\times Y)$ such that whenever $G\in\Gamma$, then $G\subset L\times Y$ for some $L\in\Lambda$.*

Definition 2. *If Q is slattable over every topological space and m and n are infinite cardinals with $n\leq m$, then Q_n (respectively Q_n^m) is the class of all topological spaces X such that, if $\mathfrak{C}$ is an open cover of X ($\mathfrak{C}$ is an open cover of X with card $(\mathfrak{C})\leqslant m$), then there exists an open refinement $\mathfrak{R}$ of $\mathfrak{C}$ and $\Gamma\in Q(X)$ with each element of Γ intersecting fewer than n elements of $\mathfrak{R}$.*

Definition 3. *The functions C, P, and M from $\mathfrak{T}$ into $\mathfrak{S}$ are defined by:*

$$C(X)=\{\{X\}\}$$
$$P(X)=\{\mathfrak{C}:\ \mathfrak{C}\text{ is an open cover of } X\}$$
$$M(X)=\{\{\{x\}: x\in X\}\}.$$

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As a simple consequence of the definition, we have the following lemma:

Lemma 1. *C, P, and M are slattable over every topological space.*

As special cases of C_n, C_n^m, P_n, P_n^m, and M_n, M_n^m (which, by Lemma 1, may be defined), note the following easily verified examples:

(i) $C_{\aleph_0}$ is the class of all compact spaces.

(ii) $C_{\aleph_1}$ is the class of all Lindelöf spaces.

(iii) $C_{\aleph_0}^{\aleph_0}$ is the class of all countably compact spaces.

(iv) $C_{\aleph_k}$ is the class of all k-compact spaces (in the sense of Erdös and Hajnal [3]).

(v) C_m is the class of all m-Lindelöf spaces (in the sense of Frolík [4]).

(vi) $C_{m'}$ is the class of all (m, ∞)-compact spaces*) (in the sense of Gál [5]).

(vii) $C_{\aleph_0}^m$ is the class of all m-compact spaces (in the sense of Frolík [4]).

(viii) $C_{m'}^n$ is the class of all (m, n)-compact spaces*) (in the sense of Gál [5]).

(ix) $P_{\aleph_0}$ is the class of all paracompact spaces.

(x) $P_{\aleph_0}^{\aleph_0}$ is the class of all countably paracompact spaces.

(xi) $P_{\aleph_0}^m$ is the class of all m-paracompact spaces (in the sense of Morita [8]).

(xii) $M_{\aleph_0}$ is the class of all metacompact spaces.

(xiii) $M_{\aleph_0}^{\aleph_0}$ is the class of all countably metacompact spaces.

3. Proof of the theorem. The following three lemmas use techniques developed by J. Dieudonné [1] and C. H. Dowker [2].

Lemma 2. *Let X be a compact topological space, Y be a topological space, and $\mathfrak{C}$ be an open cover of $X\times Y$. Then there exists an open cover $\mathfrak{D}$ of Y such that $X\times D$ is covered by finitely many sets in $\mathfrak{C}$ whenever $D\in\mathfrak{D}$.*

Proof. If $(x, y)\in X\times Y$, let M_{xy}, N_{xy}, and C_{xy} be open sets in X, Y, and $\mathfrak{C}$ respectively such that $x\in M_{xy}$, $y\in N_{xy}$, and $M_{xy}\times N_{xy}\subset C_{xy}\in\mathfrak{C}$. For each $y\in Y$, $\{M_{xy}\times N_{xy}: x\in X\}$ is an open cover of the compact space $X\times\{y\}$, so there exists a finite subset F_y of X such that $\{M_{xy}\times N_{xy}: x\in F_y\}$ is a finite open cover of $X\times\{y\}$. Let $N_y=\cap\{N_{xy}: x\in F_y\}$ and let $\mathfrak{D}=\{N_y: y\in Y\}$. Then $\mathfrak{D}$ is an open cover of Y and if $N_y\in\mathfrak{D}$, then $X\times N_y\subset\cup\{M_{xy}\times N_{xy}: x\in F_y\}\subset\cup\{C_{xy}: x\in F_y\}$.

Lemma 3. *Let X be a compact space, Y a topological space, and $\mathfrak{C}$ an open cover of $X\times Y$ with card$(\mathfrak{C})\leq m$, where m is an infinite cardinal. Then there exists an open cover $\mathfrak{D}$ of Y with*

*) Here m' is the least cardinal greater than m.

card$(\mathfrak{D}) \leqslant m$ *and for each* $D \in \mathfrak{D}$, *there exists a subcollection* $\mathfrak{C}_D$ *of* $\mathfrak{C}$ *with card*$(\mathfrak{C}_D) < m$ *such that* $\mathfrak{C}_D$ *covers* $X \times D$.

Proof. Let $\mathfrak{C} = \{C_\alpha : \alpha \in m\}$. For each $\alpha \in m$ let $S_\alpha = \bigcup\{C_\beta : \beta < \alpha\}$, $D_\alpha = \{y : y \in Y, X \times \{y\} \subset S_\alpha\}$, and $\mathfrak{D} = \{D_\alpha : \alpha \in m\}$. Then card$(\mathfrak{D}) \leqslant m$.

If $y \in Y$, then $X \times \{y\}$ is compact and thus covered by a finite subcollection $\mathfrak{F}$ of $\mathfrak{C}$. In other words, $X \times \{y\} \subset \bigcup\{C_\beta : \beta < \alpha\} = S_\alpha$, for some $\alpha \in m$; so $y \in D_\alpha$. Thus, $\mathfrak{D}$ is a cover of Y.

Also, if $D_\gamma \in \mathfrak{D}$, then $\gamma \in m$ and $X \times D_\gamma = X \times \{y : y \in Y, X \times \{y\} \subset S_\gamma\} \subset S_\gamma = \bigcup\{C_\beta : \beta < \gamma\}$. Consequently, $\mathfrak{C}_{D_\gamma} = \{C_\beta : \beta < \gamma\}$ is a subcollection of $\mathfrak{C}$ which covers $X \times D_\gamma$ and card $(\mathfrak{C}_{D_\gamma}) < m$.

Suppose $D_\alpha \in \mathfrak{D}$ and $y \in D_\alpha$. Then $X \times \{y\} \subset S_\alpha$ and S_α is open in $X \times Y$. For each $x \in X$, let M_x and N_x be open sets in X and Y respectively such that $x \in M_x$, $y \in N_x$ and $(x, y) \in M_x \times N_x \subset S_\alpha$. By the compactness of $X \times \{y\}$ there exists a finite subset F of X such that $X \times \{y\} \subset \bigcup\{M_x \times N_x : x \in F\}$. If N^y is the open set $\bigcap\{N_x : x \in F\}$, then $X \times N^y \subset \bigcup\{M_x \times N_x : x \in F\} \subset S_\alpha$. Hence $y \in N^y \subset D_\alpha$, and thus D_α is open.

Lemma 4. *Let* X *and* Y *be topological spaces,* $\mathfrak{C}$ *an open cover of* $X \times Y$, *and* m *and* n *infinite cardinals. If* $\mathfrak{D}$ *is an open cover of* Y *such that* $X \times D$ *is covered by fewer than* m *elements of* $\mathfrak{C}$ *whenever* $D \in \mathfrak{D}$ *and if* $\mathscr{R}$ *is an open refinement of* $\mathfrak{D}$, *then there exists an open refinement* $\mathfrak{R}$ *of* $\mathfrak{C}$ *such that whenever* $S \subset Y$ *and* S *intersects fewer than* n *elements of* $\mathscr{R}$, *then* $X \times S$ *intersects fewer than* $m \cdot n$ *elements of* $\mathfrak{R}$.

Proof. For each $R \in \mathscr{R}$, let $\mathfrak{C}_R$ be a subcollection of $\mathfrak{C}$ such that card $(\mathfrak{C}_R) < m$ and $\mathfrak{C}_R$ covers $X \times R$.

Let $\mathfrak{R}_R = \{(X \times R) \cap C_R : C_R \in \mathfrak{C}_R\}$ and let $\mathfrak{R} = \bigcup_{R \in \mathscr{R}} \{\mathfrak{R}_R\}$. Clearly $\mathfrak{R}$ refines $\mathfrak{C}$.

$\mathfrak{R}$ covers $X \times Y$ since if $(x, y) \in X \times Y$, then y belongs to some $R \in \mathscr{R}$ and thus $(x, y) \in X \times R$. $X \times R$ is covered by $\mathfrak{C}_R$, so $(x, y) \in C_R$ for some $C_R \in \mathfrak{C}_R$. Thus, $(x, y) \in (X \times R) \cap C_R \in \mathfrak{R}$.

$\mathfrak{R}$ is open since if $\boldsymbol{R} \in \mathfrak{R}$, $\boldsymbol{R} = (X \times R) \cap C_R$ for some $R \in \mathscr{R}$, $C_R \in \mathfrak{C}_R$. But both $X \times R$ and C_R are open in $X \times Y$.

If S is a subset of Y intersecting fewer than n sets of $\mathscr{R}$, then $X \times S$ intersects fewer than n sets of the form $X \times R$ where $R \in \mathscr{R}$ and R is one of the sets intersecting S. Since there are fewer than m sets $C_R \in \mathfrak{C}_R$ such that $(X \times R) \cap C_R$ is an element of $\mathfrak{R}$, $X \times S$ will intersect at most the fewer than m sets $(X \times R) \cap C_R$ associated with the fewer than n sets $X \times R$. Hence, $X \times S$ will intersect fewer than $m \cdot n$ sets of $\mathfrak{R}$.

Theorem. *Let* X *be a compact space and* Y *belong to* Q_n *or* Q_n^n. *Then* $X \times Y$ *belongs to* Q_n *or* Q_n^n, *respectively.*

Proof. Assume $Y \in Q_n$ or Q_n^n. Let $\mathfrak{C}$ be an arbitrary open cover of $X\times Y$.

Case (i), $Y\in Q_n$. By Lemma 2 there exists an open cover $\mathfrak{D}$ of Y such that $X\times D$ is covered by finitely many (i.e. $<\aleph_0$) sets of $\mathfrak{C}$ whenever $D\in\mathfrak{D}$.

Case (ii), $Y\in Q_n^n$. By Lemma 3 there exists a cover $\mathfrak{D}$ of Y such that card $(\mathfrak{D})\leqslant n$ and whenever $D\in\mathfrak{D}$, there exists a subcollection $\mathfrak{C}_D$ of $\mathfrak{C}$ with card $(\mathfrak{C}_D)<n$ such that $\mathfrak{C}_D$ covers $X\times D$.

In either case, extract an open refinement $\mathscr{R}$ of $\mathfrak{D}$ such that for some $\Lambda\in Q(Y)$, each element of Λ intersects fewer than n sets of $\mathscr{R}$. By Lemma 4 there exists a refinement $\mathfrak{R}$ of $\mathfrak{C}$ such that if $L\in\Lambda$, then $X\times L$ intersects fewer than $n\cdot\aleph_0=n$ or $n\cdot n=n$ sets of $\mathfrak{R}$ in cases (i) and (ii) respectively (since L intersects fewer than n sets of $\mathscr{R}$.) Since Q is slattable over Y, there exists $\Gamma\in Q(X\times Y)$ such that whenever $G\in\Gamma$, $G\subset X\times L$ for some $L\in\Lambda$. Hence each $G\in\Gamma$ intersects fewer than n sets of $\mathfrak{R}$. Thus, $\mathfrak{R}$ is the required refinement.

The following corollaries are an indication of the type of results that follow immediately from the theorem.

Corollary 1. *If X and Y are compact spaces, then $X\times Y$ is compact* [9].

Corollary 2. *If X is compact and Y is paracompact, then $X\times Y$ is paracompact* [1].

Corollary 3. *If X is compact and Y is countably paracompact, then $X\times Y$ is countably paracompact* [2].

Corollary 4. *If X is compact and Y is (m,∞)-compact, then $X\times Y$ is (m,∞)-compact* [5].

References

[1] J. Dieudonné: Une généralisation des espaces compacts. J. Math. Pures Appl., **23**, 65-76 (1944).

[2] C. H. Dowker: On countable paracompactness. Can. J. Math., **3**, 219-224 (1951).

[3] P. Erdös and H. Hajnal: On the topological product of discrete λ-compact spaces. General topology and its Relations to Modern Analysis and Algebra. Prague: Nakladatelstvi Československé Akademie Věd, 148-151 (1962).

[4] Z. Frolík: Generalisations of sompact and Lindelöf spaces. Czech. Math. J., **9**, 172-217 (1959).

[5] I. Gál: On a generalized notion of compactness I, II. Proc. Nederl. Akad. Wetensch., **60**, 421-435 (1957).

[6] Y. Hayashi: On countably metacompact spaces. Bull. Univ. Osaka Prefecture, Ser. A, **8**, 161-164 (1959-1960).

[7] J. L. Kelley: General Topology. New York. Van Nostrand (1955).

[8] K. Morita: Paracompactness and product spaces. Fund. Math., **50**, 223-236 (1961).

[9] A. Tychonoff: Über die topologische Erweiterung von Raümen. Math. Annalen, **102**, 544-561 (1930).