209. Note on Inverse Images under Open Finite-to-One Mappings

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1. Introduction and theorems. Recently, A. Arhangel'skii [2] proved the following result:

A completely regular T_2 space which is the inverse image of a metric space under an open-closed finite-to-one mapping[1] *is metrizable.* Also, in the same paper he showed that *the inverse image of a compact metric space under an open finite-to-one mapping needs not be metrizable.*[2]

Hence, we shall consider the metrizability of it adding some assumptions and obtain the following result:

Theorem 1. *If f is an open finite-to-one mapping of a normal, locally compact T_2 space X onto a metric space Y, then X is metrizable.*

On the other hand, in [8] we introduced and discussed the notion of spaces with σ-locally finite nets[3] as a class of topological spaces containing all metric spaces. As for the space with a σ-locally finite net, the following holds:

Theorem 2. *Let f be an open finite-to-one mapping of a normal T_2 space X onto a collectionwise normal T_2 space with a σ-locally finite net. Then X has a σ-locally finite net.*

If we combine Theorem 2 with the notion of M-space (cf. [7]), we can obtain the another proof of the above Arhangel'skii's theorem and a generalization of it:

Theorem 3. *Let f be an open finite-to-one mapping of a normal T_2 space X onto a collectionwise normal T_2 space Y with a σ-locally finite net and g a closed mapping of X onto a metric space Z such that $g^{-1}(z)$ is countably compact for each $z \in Z$. Then X is metrizable.*

In the following we shall prove Theorems 2, 1, and 3 using some lemmas, and construct an example of *a non-metrizable hereditarily*

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1) In this note we consider only continuous mapping.

2) The description of his example seems to contain some inaccuracies.

3) A collection $\mathfrak{B}$ of (not necessarily open) sets of a topological space X is called a *net* for X if, whenever $x \in U$ with x a point and U open in X, then $x \in B \subset U$ for some $B \in \mathfrak{B}$ (cf. [6], [3]). A net which is a union of countably many locally finite collections is called a *σ-locally finite net* (cf. [8]).

paracompact space which is the inverse image of a compact metric space under an open, order $\leqq 2$[4] *mapping.*

2. Lemmas. Lemma 1. *Let f be an open mapping of a locally compact space X onto a T_2 space Y. Then Y is also locally compact.*

Lemma 2. *Let X be a countable union of subspaces of X, each of which is Lindelöf. Then X is also Lindelöf.*

Since these two lemmas are almost clear, we omit the proofs.

The following is due to Arhangel'skii [1].

Lemma 3. *Let f be an open finite-to-one mapping of a T_2 space X onto a T_2 space Y and $Y_n=\{y \mid y \in Y, |f^{-1}(y)|=n\}$, $X_n=f^{-1}(Y_n)$ for $n=1, 2, \cdots$. Then $f_n=f \mid X_n$ is a locally homeomorphic, perfect*[5] *mapping of X_n onto Y_n.*

3. Proofs. *Proof of Theorem* 2. Let us put Y_n, X_n, and f_n as in Lemma 3 for $n=1, 2, \cdots$. Since Y is hereditarily paracompact (cf. [8], Theorem 2.9) and Y_n has also a σ-locally finite net for $n=1, 2, \cdots$ (cf. [8] Theorem 2.1), Y_n is a paracompact space with a σ-locally finite net for $n=1, 2, \cdots$. Since f_n is a locally homeomorphic, perfect mapping by Lemma 3, X_n is also a paracompact space with a σ-locally finite net $\mathfrak{B}^n=\bigcup_{m=1}^{\infty}\mathfrak{B}^n$ (cf. [8], Theorem 2.5), where we can assume $\mathfrak{B}^n_m \subset \mathfrak{B}^n_{m+1}$ for $m=1, 2, \cdots$. Let $Y'_n=\bigcup_{i=1}^{n} Y_i$ and $X'_n=f^{-1}(Y'_n)$. Then Y'_n is closed in Y. Since Y is perfectly normal (cf. [8], Theorem 2.8), we have $Y'_n=\bigcap_{i=1}^{\infty} G^n_i$ where G^n_i is an open set of Y such that $G^n_i \supset G^n_{i+1}$ for $i=1, 2, \cdots$. Put $H^n_i=f^{-1}(G^n_i)$ for $i=1, 2, \cdots$. Then $X'_n=\bigcap_{i=1}^{\infty} H^n_i$. Now we put $\mathfrak{C}^n_m=\mathfrak{B}^n_m \cap (X-H^{n-1}_m)$ for $m=1, 2, \cdots; n=1, 2, \cdots$ where $H^0_m=\phi$ for $m=1, 2, \cdots$, and $\mathfrak{C}=\bigcup_{m,n=1}^{\infty}\mathfrak{C}^n_m$. Then $\mathfrak{C}$ is a σ-locally finite net for X. That is; since H^{n-1}_m is open in X and X'_n is closed in X and, moreover, $\mathfrak{B}^n_m$ is locally finite in X'_n, $\mathfrak{C}^n_m$ is a locally finite collection in X for $m=1, 2, \cdots$, $n=1, 2, \cdots$. Therefore, $\mathfrak{C}$ is a σ-locally finite collection in X. For an arbitrary point x of X and an arbitrary open set U of X containing x let n be the smallest number such as $x \in X'_k$ and m the smallest number such as $x \notin H^{n-1}_m$. Then $x \in X_n$. Since $\mathfrak{B}^n$ is a net for X_n, there is an l such that $x \in B \subset U \cap X_n$ for some $B \in \mathfrak{B}^m_l$. Put $k=\max\{m, l\}$. Then we have $x \in (B-H^{n-1}_k) \subset U$ by the assumptions that $\mathfrak{B}^n_m \subset \mathfrak{B}^n_k$ and $H^{n-1}_m \supset H^{n-1}_k$. This shows that $\mathfrak{C}$ is a net for X, completing the proof of Theorem 2.

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4) *Order* $\leqq 2$ of a mapping $f: X \to Y$ means $|f^{-1}(y)| \leqq 2$ for each $y \in Y$, where $|f^{-1}(y)|$ is a cardinal number of $f^{-1}(y)$.

5) A closed mapping f of a space X onto a space Y is called *perfect* if $f^{-1}(y)$ is compact for each $y \in Y$.

Proof of Theorem 1. Since Y is a locally compact metric space by Lemma 1, there exists a discrete collection $\{Y_\alpha \mid \alpha \in \mathfrak{A}\}$ which is a closed covering of Y and, each of which is a countable union of compact metric subspaces $K_{\alpha n}$ ($n=1, 2, \cdots$), therefore, separable metric subspaces. When we put $X_\alpha = f^{-1}(Y_\alpha)$ for each $\alpha \in \mathfrak{A}$, we have that X is a discrete sum of $\{X_\alpha \mid \alpha \in \mathfrak{A}\}$. Hence, it is sufficient to show that each X_α is metrizable. Now, let α be a fixed element of $\mathfrak{A}$. Since $f \mid X_\alpha$ is an open finite-to-one mapping of X_α onto Y_α, by Lemma 3 we have that $Y_\alpha = \bigcup_{n=1}^{\infty} Y_{\alpha n}$, $X_\alpha = \bigcup_{n=1}^{\infty} X_{\alpha n}$ and $f \mid X_{\alpha n}$ is a locally homeomorphic, perfect mapping. Since $Y_{\alpha n}$ is separable metric and $f \mid X_{\alpha n}$ is perfect for $n=1, 2, \cdots$, $X_{\alpha n}$ is a Lindelöf space for $n = 1, 2 \cdots$ (cf. [4]). Hence, X_α is a Lindelöf space by Lemma 2, therefore, paracompact space (cf. [5]). Since a compact space is an M-space and X_α is locally compact, X_α is a paracompact, locally M-space and, moreover, a space with a σ-locally finite net by Theorem 2. Therefore, X_α is metrizable (cf. [8], Theorem 3.7), completing the proof of Theorem 1.

Proof of Theorem 3. Using f, X is a space with a σ-locally finite net by Theorem 2. Using g, X is an M-space (cf. [7]). Hence, X is a normal T_2 M-space with a σ-locally finite net, therefore, metrizable (cf. [8], Theorem 3.6).

4. Example. Let $A_0, A_1, A_2, \cdots$ be subsets of Euclidean plane R^2 such that $A_0 = \{(x, y) \mid -1 \leqq x, y \leqq 0\}$ and

$$A_n = \{(0,0)\} \cup \left\{(x, y) \,\middle|\, 0<x,\ y<1,\ \frac{1}{2n+1}x < y < \frac{1}{2n}x\right\}$$

for $n=1, 2, \cdots$, and $X = \bigcup_{n=0}^{\infty} A_n$. Let us define the topology of X as follows: G is open in X if and only if $G \cap A_n$ is open in A_n as a subspace of R^2 for each n. Since X does not satisfy the first countable axiom at $(0, 0)$, X is not matirzable. Let $Y = A_0$ be a subspace of R^2 and f a mapping of X onto Y such that

$$f((x, y)) = \begin{cases} (x, y) & \text{if } (x, y) \in A_0 \\ (-x, y) & \text{if } (x, y) \in \bigcup_{n=1}^{\infty} A_n . \end{cases}$$

Then it is easily seen that f is an open, order$\leqq 2$ mapping of a non-metrizable, hereditarily paracompact space X onto a compact metric space Y.

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