208. An Abstract Integral

By Yôto KUBOTA
Department of Mathematics, Ibaraki University

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1. Introduction. In the previous paper [2] the author considered an abstract treatment of derivatives and integrals of Perron's type. S. Izumi [1] has given an abstract consideration of the general and special Denjoy integrals using the lemma of P. Romanovski.

The aim of this paper is to extend and modify Izumi's idea and to obtain a more comphrehensive abstract integral which contains the approximately continuous Denjoy integral (AD-integral) defined by the author [3] as a special case.

2. Derivative and absolute continuity in abstract sense. Let $f(x)$ be a real valued function defined on the interval $I=[a, b]$ and α, β be real constants. We call an operation $abDf(x)$ *abstract derivative* of $f(x)$ at x provided that

(i) if $f(x)$ is differentiable at x in the ordinary sense then

$$abDf(x)=f'(x);$$

(ii) $$abD(\alpha f(x)+\beta g(x))=\alpha abDf(x)+\beta abDg(x).$$

A real valued function $F(x)$ is said to be *absolutely continuous in abstract sense* on the set E, written by $F\in ab\,AC_E$, if the following conditions are satisfied.

(iii) If $F\in ab\,AC_E$ and $E'\subset E$ then $F\in ab\,AC_{E'}$.

(iv) If $F, G\in ab\,AC_E$ then $\alpha F+\beta G\in ab\,AC_E$.

(v) If F is absolutely continuous in the ordinary sense on E then $F\in ab\,AC_E$.

(vi) If $F\in ab\,AC_E$ and E is closed then $abDF(x)$ exists at almost all points of E.

(vii) If $F(x)$ is approximately continuous on (a, b) and is non-decreasing on each complementary interval of closed set E with respect to (a, b) and if $F\in ab\,AC_E$ and $abDF(x)\geqq 0$ a.e. on E then $F(x)$ is non-decreasing on (a, b).

A finite function $F(x)$ is said to be *generalized absolutely continuous in abstract sense* on $I=[a, b]$, symbolically $F\in ab\,ACG_I$, if the interval I is the sum of a countable number of closed sets E_k $(k=1, 2\cdots)$ such that $F\in ab\,AC_{E_k}$.

3. Abstract integral. Lemma 1. *If a non-void closed set E is the sum of a countable number of closed sets E_k $(k=1, 2\cdots)$, then there exists an interval (l, m) containing points of E and an* *integer n such that* $(l, m)\cdot E \subset E_n$.

For the proof, see for example [4], p. 143.

The next lemma, due to P. Romanovski, plays an essential role in our theory.

Lemma 2 ([5], p. 543). *Let* $\boldsymbol{F}$ *be a family of open intervals in the open interval* $I_0=(a, b)$ *such that*

(i) *if* $I_k \in \boldsymbol{F}$ $(k=1, 2, \cdots, n)$ *and if* $(\bigcup_{k=1}^{n} \bar{I}_k)^\circ = I$ *is an open interval then* $I \in \boldsymbol{F}$;

(ii) *if* $I \in \boldsymbol{F}$ *and* $I' \subset I$ *then* $I' \in \boldsymbol{F}$;

(iii) *if* $\bar{I}' \subset I$ *implies* $I' \in \boldsymbol{F}$ *then* $I \in \boldsymbol{F}$;

(iv) *if* $\boldsymbol{F}_1$ *is a subfamily of* $\boldsymbol{F}$ *such that* $\boldsymbol{F}_1$ *does not cover* I_0, *then there is an* $I \in \boldsymbol{F}$ *such that* $\boldsymbol{F}_1$ *does not cover* I.

Then $I_0 \in \boldsymbol{F}$.

Theorem 1. *If* $f(x)$ *is approximately continuous on* $I=[a, b]$, $f \in ab\,ACG_I$ *and if* $abDf(x) \geqq 0$ a.e. *on* I *then* $f(x)$ *is non-decreasing on* I.

Proof. First we observe that if $f(x)$ is approximately continuous on $[a, b]$ and is non-decreasing on (a, b) then $f(x)$ is non-decreasing on $[a, b]$. Let $\boldsymbol{F}$ be the system of all open intervals of (a, b) in which f is non-decreasing. If we show that the family $\boldsymbol{F}$ satisfies the conditions of Lemma 2, then f is non-decreasing on (a, b) by Lemma 2. Hence f is also so on $[a, b]$ for the reason mentioned above.

Evidently $\boldsymbol{F}$ satisfies the conditions (i), (ii), and (iii) of Lemma 2.

Let $\boldsymbol{F}_1$ be a subfamily of $\boldsymbol{F}$ such that $\boldsymbol{F}_1$ does not cover I_0 and E be the set of points not covered by $\boldsymbol{F}_1$. Then E is clearly closed. Since $f(x)$ is $ab\,ACG_I$, the interval I is the sum of a countable number of closed sets E_k $(k=1, 2, \cdots)$ on each of which f is $ab\,AC_{E_k}$, so that we have $E=\bigcup_{k=1}^{\infty} E\cdot E_k$. It follows from Lemma 1 that there exists an interval (l, m) and an integer n such that

$$(l, m)\cdot E \subset E_n.$$

Hence f is $ab\,AC_G$ for $G=\overline{(l, m)\cdot E}$ by axiom (iii). Since f is non-decreasing on each complementary interval of G with respect to (l, m) and since $f \in ab\,AC_G$ and $abDf(x) \geqq 0$ at almost all points of G, f is non-decreasing on (l, m) by axiom (vii), and hence $(l, m) \in F$. But $(l, m) \in \boldsymbol{F}_1$, for it contains points of E. Thus the condition (iv) of Lemma 2 is satisfied, and the theorem is proved.

Corollary. *If* $f(x)$ *is approximately continuous on* $I=[a, b]$ *and* $f \in ab\,ACG_I$ *and if* $abDf(x)=0$ a.e. *then* $f(x)$ *is constant on* I.

A extended real valued function $f(x)$ defined on $[a, b]$ is said to be *Denjoy integrable in abstract sense* on $[a, b]$ or abD-integrable if there exists a function $F(x)$ which is approximately continuous, $ab\,ACG$ on $[a, b]$ and $abD\,F(x)=f(x)$ a.e. The function $F(x)$ is called an indefinite integral of $f(x)$, and the definite integral of $f(x)$ on $[a, b]$, denoted by $(abD)\int_a^b f(t)dt$, is defined as $F(b)-F(a)$.

Uniqueness of the definite integral follows from Corollary of Theorem 1.

Theorem 2. *If $f(x)$ and $g(x)$ are abD-integrable on $[a, b]$ then $\alpha f(x)+\beta g(x)$ is also so and we have*

$$(abD)\int_a^b(\alpha f+\beta g)dt=\alpha(abD)\int_a^b f\,dt+\beta(abD)\int_a^b g\,dt.$$

Proof. Since f and g are abD-integrable on $[a, b]$, there exists $F(x)$ and $G(x)$ which are both approximately continuous, $ab\,ACG$ on $[a, b]$ and satisfies the relations $abD\,F(x)=f(x)$ a.e. and $abD\,G(x)=g(x)$ a.e. The function $\alpha F+\beta G$ is approximately continuous. It is also $ab\,ACG$ on $[a, b]$ by axiom (iv) and

$$abD(\alpha F(x)+\beta G(x))=\alpha f(x)+\beta g(x)$$

by axiom (ii). Hence $\alpha f+\beta g$ is abD-integrable on $[a, b]$ and

$$(abD)\int_a^b(\alpha f+\beta g)dt=\alpha F(b)+\beta G(b)-(\alpha F(a)+\beta G(a))$$
$$=\alpha(abD)\int_a^b f\,dt+\beta(abD)\int_a^b g\,dt.$$

Theorem 3. *If $f(x)$ is abD-integrable on $[a, b]$ and $f(x)\geqq 0$ a.e. then $f(x)$ is L-integrable and*

$$(abD)\int_a^b f(t)dt=(L)\int_a^b f(t)dt.$$

Proof. Since f is abD-integrable on $[a, b]$, there exists a function $F(x)$ which is approximately continuous, $ab\,ACG$ on $[a, b]$ and $abD\,F(x)=f(x)$ a.e. Hence we have

$$abD\,F(x)=f(x)\geqq 0 \quad \text{a.e.}$$

It follows from Theorem 1 that $F(x)$ is non-decreasing on $[a, b]$, so that $F'(x)$ is L-integrable. By axiom (i), $abD\,F(x)=F'(x)=f(x)$ a.e. and therefore $f(x)$ is L-integrable on $[a, b]$. The identity

$$(abD)\int_a^b f(t)dt=(L)\int_a^b f(t)dt$$

follows from axioms (i) and (v).

Theorem 4. *If $\{f_n(x)\}$ be a convergent sequence of abD-integrable functions on $[a, b]$ such that $g(x)\leqq f_n(x)\leqq h(x)$, $g(x)$ and $h(x)$ being abD-integrable, then the limit function $f(x)=\lim f_n(x)$ is abD-integrable and*

$$\lim\,(abD)\int_a^b f_n(t)dt=(abD)\int_a^b f(t)dt.$$

Proof. Since f_n-g and $h-g$ are abD-integrable and non-negative, they are L-integrable by Theorem 3. We have clearly

$\lim (f_n - g) = f - g$ and $f_n - g \leqq h - g$. It follows from Dominated convergence theorem for Lebesgue integral that $f-g$ is L-integrable and

$$\lim (L)\int_a^b (f_n - g)dt = (L)\int_a^b (f-g)dt.$$

Hence we get from Theorems 2 and 3 that the function $f(x)$ is abD-integrable and that

$$\lim (abD)\int_a^b f_n(t)dt = (abD)\int_a^b f(t)dt.$$

The author [3] has defined the AD-integral as follows. A function $f(x)$ is said to be AD-integrable on $[a, b]$ if there exists an approximately continuous function $F(x)$ which is (ACG) on $[a, b]$ and $AD\,F(x) = f(x)$ a.e., where $F(x)$ is termed (ACG) on $[a, b]$ when the interval $[a, b]$ is the sum of a countable number of closed sets on each of which $F(x)$ is absolutely continuous in ordinary sense. The definite AD-integral $(AD)\int_a^b f(t)dt$ is defined as $F(b) - F(a)$.

If we take $ab\,ACG$ as (ACG) and abD as AD then the abD-integral becomes the AD-integral. Also the general Denjoy integral is obtained if we take $ab\,ACG$ and abD as ACG and AD respectively; because a function which is ACG on $[a, b]$ is necessarily (ACG) on $[a, b]$ for the continuity.

We can deduce further properties of the (abD)-integral in usual way when we add some axioms to the system of axioms (i)—(vii).

References

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