206. On the Sets of Points in the Ranked Space

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(Comm. by Kinjirô KUNUGI, M.J.A., Dec. 12, 1967)

1. In this section we will define the several notions, m-open sets, m-closed sets and m-accumulation points of a subset in the ranked space [1], and we will prove some propositions in respect of these notions. We have used the same terminology as that introduced in the paper "On an Equivalence of Convergences in Ranked Spaces" [6].

Definition 1. *A subset A of a ranked space R is m-open if and only if for any point p of A there is a neighborhood $V_\alpha(p)$ of p with a rank γ_α such that $V_\alpha(p)\subseteq A$. A subset A is m-closed if and only if $R-A$ is m-open.*

Definition 2. *A point p is a m-accumulation point of a subset A of a ranked space R if and only if every neighborhood of p with any rank contains points of A other than p.*

Proposition 1. *If R is a ranked space, then the following conditions are equivalent.*

(*a*) *A subset A of R is m-closed.*

(*b*) *A subset A of R contains the set consisting of its m-accumulation points.*

Proof. To prove that (a) implies (b).

Let p be a m-accumulation point of A. If $p\notin A$ then $p\in R-A$. Since A is m-closed $R-A$ is m-open. Therefore there is some neighborhood $U(p)$ of p with a rank such that $U(p)\subseteq R-A$. Hence p is not a m-accumulation point of A.

To prove that (b) implies (a).

If A is not m-closed then $R-A$ is not m-open. Therefore there is a point p belonging to $R-A$ such that every neighborhood of p with any rank intersects A. Hence p is a m-accumulation point of A and does not belong to A.

Proposition 2. *If R is a ranked space, then the conditions below are related as follows. For all space* (*a*) *implies* (*b*). *If R satisfies the following condition* (*M*):

(*M*) *if* $V(p)\in\mathfrak{V}_\alpha$, $U(p)\in\mathfrak{V}_\beta$, *and* $\alpha\geq\beta$ *then* $V(p)\subseteq U(p)$,

then (*b*) *implies* (*a*).

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(a) *A point p is a m-accumulation point of a subset A of R.*

(b) *There is a sequence in $A-\{p\}$ which R-converges to p.*

Proof. To prove that (a) implies (b).

Since R is a ranked space, there is a fundamental sequence [2] $\{V_\alpha(p)\}$ of neighborhoods of p. Since p is a m-accumulation point of A, for each α there is a point p_α such that $p_\alpha \in V_\alpha(p) \cap (A-\{p\})$. Hence the sequence $\{p_\alpha\}$ satisfies (b).

To prove that (b) implies (a) when (M) is satisfied.

Let $\{p_\alpha\}$ be a sequence in $A-\{p\}$ which R-converges to p. Then there is a fundamental sequence $\{V_\alpha(p)\}$ of neighborhoods of p such that $p_\alpha \in V_\alpha(p)$ and $p_\alpha \in A-\{p\}$. Let $U(p)$ be a neighborhood of p with a rank. By the condition (M), there is $V_{\alpha_0}(p)$ in the fundamental sequence such that $U(p) \supseteq V_{\alpha_0}(p)$. Since $P_{\alpha_0} \in V_{\alpha_0}(p)$ and $p_{\alpha_0} \in A-\{p\}$, p is a m-accumulation point of A.

Proposition 3. *If R is a ranked space then the conditions below are related as follows. For all spaces* (a) *implies* (b). *If R satisfies the following condition* (M):

(M) *if* $V(p) \in \mathfrak{B}_\alpha$, $U(p) \in \mathfrak{B}_\beta$, *and* $\alpha \geq \beta$ *then* $V(p) \subseteq U(p)$,

then (b) *implies* (a).

(a) *Each sequence which R-converges to a point of a subset A of R is eventually* [4] *in A.*

(b) *A subset A of R is m-open.*

Proof. To prove that (a) implies (b).

If A is not m-open, then there is a point q belonging to A such that every neighborhood of q with any rank intersects $R-A$. Since R is a ranked space there is a fundamental sequence $\{V_\alpha(q)\}$ of neighborhoods of q. Consequently, for every α there is a point q_α such that $q_\alpha \in V_\alpha(q) \cap (R-A)$. Hence the sequence $\{q_\alpha\}$ is not eventually in A.

To prove that (b) implies (a) when (M) is satisfied.

Let $\{p_\alpha\}$ be a sequence that R-converges to a point p of A. Then there is a fundamental sequence $\{V_\alpha(p)\}$ of neighborhoods of p such that $p_\alpha \in V_\alpha(p)$. Since A is m-open and $p \in A$ there is a neighborhood $V(p)$ of p with a rank γ such that $V(p) \subseteq A$. By the condition (M), for all γ_α that is larger than γ we have $V(p) \supseteq V_\alpha(p)$. Since $p_\alpha \in V_\alpha(p)$, $\{p_\alpha\}$ is eventually in A.

Remark 1. In a usual topological space, propositions [5] corresponding to our Proposition 2 and Proposition 3 are proved under the first axiom of countability. But, this condition is not necessary in the ranked space.

Remark 2. In the Proposition 2, it is proved that (b) implies (a) under the condition (M). The following example shows that some additional conditions are necessary to prove (b) implies (a).

Example 1. Let E be the unit circle in the complex plan such that $|z| \leq 1$. According to K. Kunugi [2], for each point p belonging to E we define a neighborhood of p with a rank as follows.

(1) If $|p| = 1$, we consider the set $V(\varepsilon, \delta; p)$ of all points z of E each of which satisfies the following conditions:

1) $|z-p| < \varepsilon$,

2) if $z \neq p$, $|\arg(p-z) - \arg p| < \frac{\pi}{2} - \delta$, where $\varepsilon > 0$ and $0 < \delta < \frac{\pi}{2}$.

We say that $V(\varepsilon, \delta; p)$ is a neighborhood of p with the rank $n = \left[\frac{1}{\varepsilon}\right]$.

(2) If $|p| < 1$, we consider the set $V(\varepsilon; p)$ of all points z of E each of which satisfies the two following inequalities:

$$|z| < 1 \quad \text{and} \quad |z-p| < \varepsilon.$$

We say that $V(\varepsilon; p)$ is a neighborhood of p with the rank $n = \left[\frac{1}{\varepsilon}\right]$.

Then E is a ranked space and does not satisfy the condition (M). Now, let $|p| = 1$, $0 < \theta < \frac{\pi}{2}$ and $\{z_n\}$ be a sequence of points such that

$|z_n| < 1$, $|z_n - p| = 2|z_{n+1} - p|$, and $\arg(p - z_n) - \arg p = \frac{\pi}{2} - \theta$ for $n = 1, 2, 3, \cdots$. The sequence $\{z_n\}$ is R-convergent to p. However, p is not a m-accumulation point of $\{z_n\}$, because the neighborhood $V(\varepsilon, \delta; p)$ such that $\theta < \delta$ does not intersect $\{z_n\}$.

Remark 3. In the proposition 3, it is proved that (b) implies (a) under the condition (M). The following example shows that some additional conditions are necessary to prove (b) implies (a).

Example 2. Let us consider the ranked space introduced in the Example 1. Any neighborhood of p in E is m-open. Hence $V(\varepsilon, \delta; p)$ is m-open, however $\{z_n\}$ is not eventually in $V(\varepsilon, \delta; p)$.

Remark 4. Y. Yoshida gave a definition of a open set in the ranked space [3]. He called it the (r)-open set. The (r)-open set is distinct from the m-open set defined in this paper. It is true that "(r)-open" implies "m-open in our sense". The converse is not necessarily true. For example, in the Example 2, the neighborhood $V(\varepsilon, \delta; p)$ of p is m-open, but it is not (r)-open.

2. In this section we will define m-ω-accumulation points of a set and m-cluster points of a sequence in the ranked space, and we will prove a proposition in respect of these notions.

Definition 3. *A point p is a m-ω-accumulation point of a subset A of the ranked space if and only if every neighborhood of p with a rank contains infinitely many points of A.*

A m-ω-accumulation point of a set is a m-accumulation point of its set.

Definition 4. *A point p is a m-cluster point of a sequence $\{p_a\}$ in the ranked space if and only if $\{p_a\}$ is frequently* [4] *in every neighborhood of p with a rank.*

Proposition 4. *If R is a ranked space, then the conditions below are related as follows. For all spaces (a) is equivalent to (b) and (b) implies (c). If R satisfies the condition (M):*

(M) if $V(p)\in\mathfrak{B}_a$, $U(p)\in\mathfrak{B}_\beta$, and $\alpha\leq\beta$ then $V(p)\supseteq U(p)$, then all there conditions are equivalent.

(a) Every infinite subset of R has a m-ω-accumulation point.

(b) Every sequence in R has a m-cluster point.

(c) For every sequence in R there is a subsequence R-converging to a point of R.

Proof. To prove that (a) implies (b).

Let $\{p_\alpha\}$ be an arbitrary sequence in R. Either the range of $\{p_\alpha\}$ is infinite, in which case each m-ω-accumulation point of this infinite set is a m-cluster point of $\{p_\alpha\}$, or else the range of $\{p_\alpha\}$ is finite. In the latter case, for some point p of the ranked space, $p_\alpha=p$ for infinitely many natural numbers α, and p is a m-cluster point of $\{p_\alpha\}$.

To prove that (b) implies (a).

Let A be an arbitrary infinite subset of R. Then there is a sequence of distinct points in A, and each m-cluster point of such a sequence is a m-ω-accumulation point of A.

To prove that (b) implies (c).

Let $\{p_\beta\}$ be an arbitrary sequence in R. By (b), $\{p_\beta\}$ has a m-clustar point p. Consequently, for any neighborhood $V_1(p)\in\mathfrak{B}_{\gamma_1}$ of the point p and any natural number α, there is a point p_{β_1} such that $\alpha\leq\beta_1$ and $p_{\beta_1}\in V_1(p)$. Since R is a ranked space, for $V_1(p)$ and γ_1 tnere is a neighborhood $V_2(p)$ of the point p such that $V_1(p)\supseteq V_2(p)$, $V_2(p)\in\mathfrak{B}_{\gamma_2}$, and $\gamma_1\leq\gamma_2$. Since p is a m-cluster point of $\{p_\beta\}$, for $V_2(p)\in\mathfrak{B}_{\gamma_2}$ and β_1 there is a point p_{β_2} such that $\beta_1\leq\beta_2$ and $p_{\beta_2}\in V_2(p)$. Repeating the argument, we have a subsequence $\{p_{\beta_\alpha}\}$ of $\{p_\beta\}$ which is R-convergent to p.

To prove that (c) implies (b) when (M) is satisfied.

Let $\{p_\beta\}$ be an arbitrary sequence of R. Since $\{p_\beta\}$ has a subsequence $\{p_{\beta_\alpha}\}$ which is R-convergent to a point p of R, there is a fundamental sequence $\{V_\alpha(p)\}$ of neighborhoods of p such that $p_{\beta_\alpha}\in V_\alpha(p)$. Let $U(p)$ be an arbitrary neighborhood of p with a rank. By the condition (M), there is $V_{\alpha_0}(p)$ in the fundamental sequence such that $U(p)\supseteq V_{\alpha_0}(p)$. Hence $p_{\beta_{\alpha_0}}\in U(p)$. Therefore the point p is a m-cluster point of the sequence $\{p_\beta\}$.

Remark 5. In a usual topological space, the proposition corresponding to our Proposition 4 is proved under the first axiom of countability. But, this condition is not necessary in the ranked space.

References

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