203. *Some Remarks on Duality Theorems of Lie Groups*

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1. **Introduction.** T. Tannaka [1] proved a duality theorem for compact groups. Afterwards C. Chevalley [2] introduced the representative algebra $R(G)$ and the character set $\mathrm{Hom}(R(G), \boldsymbol{C})$ and proved Tannaka's theorem for compact Lie groups anew. This work of Chevalley revealed the relation between the compact Lie groups and algebraic groups. The representative algebra $R(G)$ of a general Lie group (not necessarily compact) was studied by G. Hochschild and G. D. Mostow in [3]. They give several conditions each of which is equivalent to say that $R(G)$ is finitely generated. One of these conditions says that the canonical homomorphism maps the connected component G_1 of G onto the connected component of the real proper automorphism group G^* of $R(G)$. This suggests a kind of duality theorem for G.

In this note we say that the duality theorem holds for a topological group G if the canonical homomorphism $\Psi: g \mapsto R_g$ is an isomorphism of G onto the real proper automorphism group G^* of $R(G)$ (cf. 3 for the definitions of G^* and R_g). In 4, we study the relation between our duality theorem and the Tannaka duality theorem (Theorem 1). In 5 we give a necessary and sufficient condition that a Lie group with a finite number of connected components satisfies the duality theorem (Theorem 2). Theorem 2 gives the intimate relation between the duality theorem and the algebraic group structure.

2. **The Tannaka duality theorem.** Let G be a topological group. In this note, a representation of G means a continuous homomorphism D of G into $GL(n, \boldsymbol{C})$ for some natural number n which is called the degree of D and denoted by $d(D)$. The set of all representations of G is called the dual object of G and denoted by $\mathfrak{R}$. For elements D_1, D_2, and D in $\mathfrak{R}$, the direct sum $D_1 \oplus D_2$, the tensor product $D_1 \otimes D_2$, the equivalent representation $\gamma D \gamma^{-1}$ ($\gamma \in GL(d(D), \boldsymbol{C})$) and the complex conjugate representation $\bar{D}$ are defined as usual. A complex representation ζ of $\mathfrak{R}$ is, by definition, a mapping from $\mathfrak{R}$ into $\bigcup_n GL(n, \boldsymbol{C})$ which satisfies

0) $\zeta(D) \in GL(d(D), \boldsymbol{C})$, 1) $\zeta(D_1 \oplus D_2) = \zeta(D_1) \oplus \zeta(D_2)$,

2) $\zeta(D_1 \otimes D_2) = \zeta(D_1) \otimes \zeta(D_2)$, 3) $\zeta(\gamma D \gamma^{-1}) = \gamma \zeta(D) \gamma^{-1}$

for any representations D_1, D_2, D, and any regular matrix γ of degree $d(D)$.

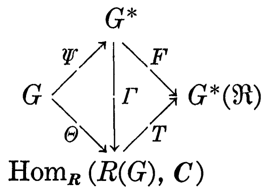
The set of all complex representations of the dual object $\mathfrak{R}$ is denoted by $G^{*C}(\mathfrak{R})$. The topology of $G^{*C}(\mathfrak{R})$ is defined as the weakest topology making the maps $\zeta \mapsto \zeta(D) \in GL(d(D), \boldsymbol{C})$ continuous for every D in $\mathfrak{R}$. Then $G^{*C}(\mathfrak{R})$ forms a topological group with the group operation defined by $\zeta_1\zeta_2^{-1}(D)=\zeta_1(D)\zeta_2(D)^{-1}$. $G^{*C}(\mathfrak{R})$ is called the complex Tannaka group of G. The subgroup $G^*(\mathfrak{R})$ of $G^{*C}(\mathfrak{R})$ defined by $G^*(\mathfrak{R})=\{\zeta \in G^{*C}(\mathfrak{R}); \zeta(\bar{D})=\overline{\zeta(D)}$ for any D in $\mathfrak{R}\}$ is called the Tannaka group of G. An element of $G^*(\mathfrak{R})$ is called a representation of $\mathfrak{R}$. For any element g in G the mapping $\zeta_g: D \mapsto D(g)$ belongs to $G^*(\mathfrak{R})$. Moreover the mapping $\Phi: g \mapsto \zeta_g$ is a continuous homomorphism of G into $G^*(\mathfrak{R})$. When this canonical homomorphism Φ is an isomorphism of G onto $G^*(\mathfrak{R})$ as topological groups, we say that the Tannaka duality theorem holds for the group G.

3. **The duality theorem.** Let G and $\mathfrak{R}$ be the same as above. The set $R(G)$ of all finite linear combinations of the matricial elements of the representations of G forms an algebra over $\boldsymbol{C}$ and is called the representative algebra of G. Any element g in G induces the right translation R_g and the left translation L_g on $R(G)$ which are defined by $(R_g f)(h)=f(hg)$ and $(L_g f)(h)=f(gh)$. An automorphism α of the algebra $R(G)$ which commutes with every left translation is called a proper automorphism. The group of all proper automorphisms of $R(G)$ is denoted by G^{*C}. The topology of G^{*C} is defined as the weakest topology making $\alpha \mapsto \lambda(\alpha(f))$ continuous for every linear form λ on $R(G)$ and every f in $R(G)$. This topology makes G^{*C} a topological group. The real proper automorphism group G^* is defined as $G^*=\{\alpha \in G^{*C}; \alpha(\bar{f})=\overline{\alpha(f)}$ for any f in $R(G)\}$. The canonical mapping $\Psi: g \mapsto R_g$ is a continuous homomorphism of G into G^*. When this canonical homomorphism Ψ is an isomorphism of G onto G^* as topological groups, we say that the duality theorem holds for the group G.

4. **The relation between two kinds of duality theorems.** The set of all homomorphisms of the algebra $R(G)$ into $\boldsymbol{C}$ which maps 1 to 1 is denoted by $\mathrm{Hom}(R(G), \boldsymbol{C})$. Let e be the identity element of G. Then any element α in G^{*C} defines a homomorphism $\omega \in \mathrm{Hom}(R(G), \boldsymbol{C})$, $\omega: f \mapsto (\alpha f)(e)$. Conversely any ω in $\mathrm{Hom}(R(G), \boldsymbol{C})$ determines a proper automorphism α by the identity $\alpha(f)(g)=\omega(L_g f)$. So the mapping $\Gamma: \alpha \mapsto \omega$ is a bijection of G^{*C} onto $\mathrm{Hom}(R(G), \boldsymbol{C})$. Γ maps the real proper automorphism group G^* onto $\mathrm{Hom}_{\boldsymbol{R}}(R(G), \boldsymbol{C}) =\{\omega \in \mathrm{Hom}(R(G), \boldsymbol{C}), \omega(\bar{f})=\overline{\omega(f)}$ for any f in $R(G)\}$. The topology of $\mathrm{Hom}(R(G), \boldsymbol{C})$ is defined as the weakest topology making $\omega \mapsto \omega(f)$

continuous for every f in $R(G)$. Then Γ is clearly continuous. To prove Γ^{-1} is also continuous, let f be an element in $R(G)$. Then the subspace $V=\{L_g f; g \in G\}_C$ is finite dimensional, so there exist a finite number of elements $f_1, \cdots, f_n$ in V and $a_1, \cdots, a_n$ in $R(G)$ such that $L_g f = \sum_i a_i(g) f_i$. Applying $\omega = \Gamma(\alpha)$ on both sides of the last equality, we get $\alpha(f)=\sum_i \omega(f_i)a_i$ and $\lambda(\alpha(f))=\sum_i \omega(f_i)\lambda(a_i)$. This proves that Γ^{-1} is continuous. So Γ is a homeomorphism of G^{*C} onto $\mathrm{Hom}\,(R(G), \boldsymbol{C})$.

Every ω in $\mathrm{Hom}\,(R(G), \boldsymbol{C})$ defines an element ζ_ω of $G^{*C}(\mathfrak{R})$ which maps D in $\mathfrak{R}$ to the matrix whose (i, j)-element is $\omega(D_{ij})$ where $D_{ij}(g)$ is the (i, j)-element of $D(g)$. So we get a continuous mapping $T: \omega \mapsto \zeta_\omega$ of $\mathrm{Hom}\,(R(G), \boldsymbol{C})$ into $G^{*C}(\mathfrak{R})$. T maps $\mathrm{Hom}_{\boldsymbol{R}}\,(R(G), \boldsymbol{C})$ into $G^*(\mathfrak{R})$. The map $T \circ \Gamma = F$ is a continuous homomorphism of G^{*C} into $G^{*C}(\mathfrak{R})$ which maps G^* into $G^*(\mathfrak{R})$. Lastly we define the continuous mapping $\Theta: g \mapsto \omega_g(\omega_g(f)=f(g))$ of G into $\mathrm{Hom}_{\boldsymbol{R}}\,(R(G), \boldsymbol{C})$. Then the following diagram is commutative.



And the canonical mapping Φ defined in 2 is the composition of the two mapping Ψ and F:

$$\Phi = F \circ \Psi \tag{1}$$

The mapping T is injective because the matricial elements of the representations span the vector space $R(G)$. So the homomorphism F is a continuous isomorphism of G^{*C} into $G^{*C}(\mathfrak{R})$. Now suppose that the homomorphism Φ induces an isomorphism of G onto $G^*(\mathfrak{R})$ as topological groups. Then, by the identity (1), the mapping $F \mid G^*$ and so the homomorphism Ψ are also topological isomorphisms. So we get the first half of the following Theorem 1.

Theorem 1. 1) *If the Tannaka duality theorem holds for a topological group G, then the duality theorem* (*defined in* 3) *holds for G.*

2) *If every representation of G is completely reducible, then the isomorphism F is surjective. In this case the duality theorem for G implies the Tannaka duality theorem of G.*

To prove the latter half of Theorem 1, we choose a representative D^α from each equivalence class α of irreducible representations of G and form the complete set of representatives $\mathfrak{D}=\{D^\alpha: \alpha \in A\}$. The set $\mathfrak{B}=\{D^\alpha_{ij}: \alpha \in A, 1 \leqq i, j \leqq d(D^\alpha)\}$ forms a basis of the vector space $R(G)$ because $\mathfrak{B}$ is linearly independent by a theorem of Burnside. So every element ζ in $G^{*C}(\mathfrak{R})$ defines uniquely a linear form ω which maps D^{α}_{ij} into the (i, j)-element of the matrix $\zeta(D^{\alpha})$. Now we shall prove that ω belongs to Hom$(R(G), \boldsymbol{C})$ and that $\zeta(D)=\zeta_{\omega}(D)$ for any D in $\mathfrak{R}$. Any representation D can be represented as $D=\gamma(D^{\alpha_1}\oplus \cdots \oplus D^{\alpha_n})\gamma^{-1}$, $(\alpha_i \in A)$ by the assumption of the completely reducibility. As ω is a linear form and $\zeta(D^{\alpha}) =(\omega(D^{\alpha}_{ij}))$, we get $\zeta(D)=(\omega(D_{ij}))$. Let D and D' be in $\mathfrak{R}$. Then we have $(\omega(D_{ij}D'_{kl}))=\zeta(D\otimes D')=\zeta(D)\otimes\zeta(D')=(\omega(D_{ij})\omega(D'_{kl}))$. This proves that ω belongs to Hom$(R(G), \boldsymbol{C})$ and $\zeta=\zeta_{\omega}$. Thus the mapping T is surjective. T is also a homeomorphism. This can be easily seen by the definition of the topologies on $G^{*C}(\mathfrak{R})$ and Hom$(R(G), \boldsymbol{C})$.

Therefore the homomorphism F is an isomorphism of G^{*C} onto $G^{*C}(\mathfrak{R})$ as topological groups. If ζ belongs to $G^{*}(\mathfrak{R})$, then ω belongs to $\mathrm{Hom}_{\boldsymbol{R}}(R(G), \boldsymbol{C})$. So F induces an isomorphism of G onto $G^{*}(\mathfrak{R})$.

In this case, if Ψ is a topological isomorphism of G onto G^{*}, then the homomorphism $\Phi=F\circ\Psi$ is a topological isomorphism of G onto $G^{*}(\mathfrak{R})$. Theorem 1 is thus proved.

5. The duality theorem for Lie groups. For a Lie group, we can give an intrinsic meaning to the duality theorem defined in 3 by the following theorem.

Theorem 2. *Let G be a Lie group with a finite number of connected components. Then G satisfies the duality theorem if and only if G is a real affine algebraic group and every (continuous) representation of G is a rational representation. When this condition is satisfied, the proper automorphism group G^{*C} of $R(G)$ can be regarded as the complexification of the real algebraic group G.*

Let G be a real affine algebraic group whose every representation is rational. Then G is a Lie group with a finite number of components. The real representative algebra $R_{\boldsymbol{R}}(G)=\{f \in R(g); \bar{f}=f\}$ contained in the affine algebra A (the algebra of everywhere defined rational functions on G) of G, because every representation is rational. On the other hand, $A\subset R_{\boldsymbol{R}}(G)$, because $A=\boldsymbol{R}[D_{11}, \cdots, D_{nn}, (\det D)^{-1}]$ for a faithful rational representation D of G. As G is a real affine algebraic set, $\Theta: g\mapsto\omega_g$ is a bijection of G onto Hom$(R_{\boldsymbol{R}}(G), \boldsymbol{R})$. On the other hand, the restriction map of $\mathrm{Hom}_{\boldsymbol{R}}(R(G), \boldsymbol{C})$ into Hom$(R_{\boldsymbol{R}}(G), \boldsymbol{R})$ is clearly a bijection. So the canonical homomorphism Ψ is a bijection of G onto G^{*}. To prove Ψ is also a homeomorphism, let $\{x_1, \cdots, x_n\}$ be a set of generators of the affine algebra $A=R_{\boldsymbol{R}}(G)$. Then the mappin $\omega\mapsto(\omega(x_1), \cdots, \omega(x_n))$ is a homeomorphism of Hom$(R_{\boldsymbol{R}}(G), \boldsymbol{R})$ onto an affine algebraic subset of $\boldsymbol{R}^n$. So Hom$(R_{\boldsymbol{R}}(G), \boldsymbol{R})$, $\mathrm{Hom}_{\boldsymbol{R}}(R(G), \boldsymbol{C})$ and G^{*} are locally compact Hausdorff spaces. On the other hand G, being a Lie group with a finite number of components, is the union of a countable number of compact sets. So the continuous isomorphism Ψ of G onto G^* is an open mapping and therefore an isomorphism as topological groups. Thus the duality theorem holds for the group G. In this case the proper automorphism group G^{*C} is the complex algebraic group with the affine algebra $R(G)$ which is the scalar extension of $R_{\boldsymbol{R}}(G)$ ($R(G)=R_{\boldsymbol{R}}(G)\otimes_{\boldsymbol{R}}\boldsymbol{C}$). So G^{*C} is the complexification of G.

Conversely, let G be a Lie group with a finite number of components for which the duality theorem holds. Let G_1 be the connected component of e in G. Then the topological isomorphism Ψ maps G_1 onto the connected component of G^*. Therefore by a theorem of G. Hochschild and G. D. Mostow [3, Theorem 7.1], $R(G)$ is finitely generated. So the algebra $R_{\boldsymbol{R}}(G)$ is also finitely generated. The duality theorem for G assures that the canonical mapping $\Theta: g\mapsto\omega_g$ is a bijection of G onto $\mathrm{Hom}(R_{\boldsymbol{R}}(G), \boldsymbol{R})$. So G is a real affine algebraic set with the affine algebra $R_{\boldsymbol{R}}(G)$. As the left translates of any element f in $R_{\boldsymbol{R}}(G)$ span a finite dimensional subspace, there are a finite number of elements $a_1, \cdots, a_n, b_1, \cdots, b_n$ in $R_{\boldsymbol{R}}(G)$ such that $f(gh)=\sum a_1(g)b_i(h)$ for any g, h in G. So the mapping $(g, h)\mapsto gh$ is regular (=everywhere defined rational) mapping. Moreover, if D belongs to $\mathfrak{R}_{\boldsymbol{R}}=\{D\in\mathfrak{R};\ \bar{D}=D\}$ then the contragredient representation $D^*(g)={}^tD(g^{-1})$ belongs to $\mathfrak{R}_{\boldsymbol{R}}$, so the mapping $g\mapsto g^{-1}$ is also regular. Thus the group G is a real affine algebraic group with the affine algebra $R_{\boldsymbol{R}}(G)$. Therefore every continuous real representation of G is a rational representation. So every (continuous) representation over $\boldsymbol{C}$ is also rational. This completes the proof of Theorem 2. Theorem 1 and 2 explain the reason why the Tannaka buality theorem of connected semisimple Lie groups obtained by Harish-Chandra [4] has a slightly weaker form than that is defined in 2 of this note. In fact, a connected semisimple Lie group is not necessarily an algebraic group.

References

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