4. Relations between Unitary ρ-Dilatations and Two Norms

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Introduction. In this paper we discuss classes of power bounded operators on a Hilbert space H and we use the notations and terminologies of [5]. Following [1] [2] [5], an operator T on H possesses a unitary ρ-dilatation if there exists a Hilbert space K containing H as a subspace, a positive constant ρ and a unitary operator U on K satisfying the following representation

$$(1) \qquad T^n=\rho\cdot PU^n \qquad (n=1, 2, \cdots)$$

where P is the orthogonal projection of K on H. Put C_ρ the class of operators, whose powers T^n admit a representation (1).

It is well known that $T\in C_1$ is characterized by $\|T\|\leqq 1$. Moreover $T\in C_2$ is characterized by $\|T\|_N\leqq 1$, where $\|T\|_N$, usually called the numerical radius of T, is defined by

$$\|T\|_N=\sup|(Th, h)| \quad \textit{for every unit vector } h \textit{ in } H.$$

The latter fact was discovered by C.A. Berger (not yet published).

Using function theoretic methods, B. Sz-Nagy and C. Foias have given a characterization of C_ρ and shown the monotonity of C_ρ as a generalization of C_1 and C_2. Hence we may naturally expect that the condition for $T\in C_\rho$ depends upon $\|T\|$ and $\|T\|_N$ together. In this paper, as a continuation of calculations in the preceding paper [3], we give a simple sufficient condition for $T\in C_\rho$ related to both $\|T\|$ and $\|T\|_N$ and its graphic expression.

1. The following theorems are known.

Theorem A ([5]). *An operator T in H belongs to the class C_ρ if and only if it satisfies the following conditions*:

$$(\mathrm{i}) \quad \begin{cases} (I_\rho) \quad \|h\|^2-2\left(1-\dfrac{1}{\rho}\right)\mathrm{Re}(zTh, h)+\left(1-\dfrac{2}{\rho}\right)\|zTh\|^2\geqq 0 \\ \qquad \qquad \textit{for } h \textit{ in } H \textit{ and } |z|\geqq 1, \\ (II) \quad \textit{the spectrum of } T \textit{ lies in the closed unit disk.} \end{cases}$$

(ii) *If $\rho\leqq 2$, then the conditon (I_ρ) implies (II).*

Theorem B ([5]). *C_ρ is non-decreasing with respect to the index ρ in the sense that*

$$C_{\rho_1}\subset C_{\rho_2} \qquad \textit{if } 0\leqq\rho_1<\rho_2.$$

Theorem C ([1]).

(i) $\begin{cases} \text{If } \|T\| \leqq \dfrac{\rho}{2-\rho} \text{ and } 0 \leqq \rho \leqq 1, \text{ then } T \in C_\rho. \\ \text{If } \|T\| \leqq 1, \text{ then } T \in C_\rho \text{ for } \rho \geqq 1. \end{cases}$

(ii) $\begin{cases} \text{If } T \in C_\rho \text{ for } 0 \leqq \rho \leqq 1, \text{ then } r(T) \leqq \dfrac{\rho}{2-\rho}. \\ \text{If } T \in C_\rho \text{ for } \rho \geqq 1, \text{ then } r(T) \leqq 1. \end{cases}$

where $r(T)$ means the spectral radius of T.

An operator T is called to be normaloid if $\|T\| = \|T\|_N$ or equivalently the spectral radius is equal to $\|T\|$ ([4]).

Theorem D ([1][3]). *If T is normaloid, $T \in C_\rho$ if and only if*

$$\|T\| \leqq \begin{cases} \dfrac{\rho}{2-\rho} & \text{if } 0 \leqq \rho \leqq 1 \\ 1 & \text{if } \rho \geqq 1 \end{cases}.$$

Theorem D was proved by E. Durszt for normal operators and by C. A. Berger and J. G. Stampfli ([1]). The author has given a simplified proof of the same theorem in [3] independently.

2. For $0 \leqq \rho \leqq 2$, the condition (I_ρ) is replaced by

$$(2-\rho)\|zTh\|^2 - 2(1-\rho)\mathrm{Re}(zTh, h) - \rho\|h\|^2 \leqq 0 \qquad \text{for } h \in H.$$

That is,

$$(I_\rho') \qquad (2-\rho)\|Th\|^2 r^2 - 2(1-\rho)|(Th, h)|\, r \cdot \cos\psi - \rho \leqq 0$$

for every unit vector h in H, where $z = re^{i\theta}$, $0 \leqq r \leqq 1$, $\psi = \varphi + \theta$ and φ is the argument of (Th, h). Since the left-hand side of (I_ρ') is negative for r $(0 \leqq r \leqq 1)$ if it is so at $r = 1$, (I_ρ') is equivalent to

$$(I_\rho'') \qquad (2-\rho)\|Th\|^2 - 2(1-\rho)|(Th, h)|\cos\psi - \rho \leqq 0$$

for every unit vector h in H.

Theorem 1. *(I_ρ) implies* $\|T\|_N \leqq \begin{cases} \dfrac{\rho}{2-\rho} & \text{if } 0 \leqq \rho \leqq 1 \\ 1 & \text{if } 1 \leqq \rho \leqq 2. \end{cases}$

Proof. Let $0 \leqq \rho \leqq 1$. By (I_ρ''), (I_ρ) is equivalent to

$$F_1(\rho, h) \equiv (2-\rho)|Th\|^2 + 2(1-\rho)|(Th, h)| - \rho \leqq 0$$

for every unit vector h in H. That is

$$(I_\rho) \text{ is true if and only if } \sup_{\|h\|=1} F_1(\rho, h) \leqq 0.$$

The following inequality is clear

$$(*) \quad (2-\rho)\|T\|_N^2 + 2(1-\rho)\|T\|_N - \rho \leqq \sup_{\|h\|=1} F_1(\rho, h) \leqq (2-\rho)\|T\|^2 + 2(1-\rho)\|T\|_N - \rho \leqq (2-\rho)\|T\|^2 + 2(1-\rho)\|T\| - \rho.$$

Consequently (I_ρ) implies

$$(2-\rho)\|T\|_N^2 + 2(1-\rho)\|T\|_N - \rho \leqq 0,$$
$$(\|T\|_N + 1) \cdot \{(2-\rho)\|T\|_N - \rho\} \leqq 0.$$

Hence

$$\|T\|_N \leqq \frac{\rho}{2-\rho}.$$

Now let $1 \leqq \rho \leqq 2$, then the condition (I_ρ'') is equivalent to

$$F_2(\rho, h) \equiv (2-\rho)\,\|Th\|^2 + 2(\rho-1)\,|(Th, h)| - \rho \leqq 0$$

for every unit vector h in H. That is

(I_ρ) *is true if and only if* $\sup_{\|h\|=1} F_2(\rho, h) \leqq 0$.

The following inequality is also clear.

$$(**)\quad (2-\rho)\,\|T\|_N^2 + 2(\rho-1)\,\|T\|_N - \rho \leqq \sup_{\|h\|=1} F_2(\rho, h) \leqq (2-\rho)\,\|T\|^2 + 2(\rho-1)\,\|T\|_N - \rho \leqq (2-\rho)\,\|T\|^2 + 2(\rho-1)\,\|T\| - \rho.$$

Consequently (I_ρ) implies

$$(2-\rho)\,\|T\|_N^2 + 2(\rho-1)\,\|T\|_N - \rho \leqq 0,$$
$$(\|T\|_N - 1)\{(2-\rho)\,\|T\|_N + \rho\} \leqq 0.$$

Hence

$$\|T\|_N \leqq 1$$ q.e.d.

Theorem 1 gives a precise limitation of $\|T\|_N$ for $T \in C_\rho$. Since $r(T) \leqq \|T\|_N$ ([4]) we get immediately.

Corollary 1 ([5]). *For* $\rho \leqq 2$, (I_ρ) *implies* (II).

C. A. Berger has characterized $T \in C_2$ by $\|T\|_N \leqq 1$. This fact and the monotonity of C_ρ give the corollary 1. But in our method the estimation of $\|T\|_N$ comes to give the proof without complicated calculations. Moreover by $(*)$ and $(**)$ in the proof of Theorem 1 we can sharpen Theorem C and give a simple sufficient condition for $T \in C_\rho$ as shown in the next section.

3. The following theorems are obvious by Theorem 1 and inequalities $(*)$, $(**)$.

Theorem 2. (i) *For* $0 \leqq \rho \leqq 1$. $T \in C_\rho$ *if and only if* $\sup_{\|h\|=1} F_1(\rho, h) \leqq 0$. (ii) *For* $1 \leqq \rho \leqq 2$. $T \in C_\rho$ *if and only if* $\sup_{\|h\|=1} F_2(\rho, h) \leqq 0$.

Theorem 3. (i) *For* $0 \leqq \rho \leqq 1$. *If* $T \in C_\rho$, *then* $\|T\|_N \leqq \dfrac{\rho}{2-\rho}$.

(ii) *For* $1 \leqq \rho \leqq 2$. *If* $T \in C_\rho$, *then* $\|T\|_N \leqq 1$.

Theorem 4. (i) *For* $0 \leqq \rho \leqq 1$. *If* $(2-\rho)\,\|T\|^2 + 2(1-\rho)\,\|T\|_N - \rho \leqq 0$, *then* $T \in C_\rho$.

(ii) *For* $1 \leqq \rho \leqq 2$. *If* $(2-\rho)\,\|T\|^2 + 2(\rho-1)\,\|T\|_N - \rho \leqq 0$, *then* $T \in C_\rho$.

Corollary 2 ([1]). (i) *For* $0 \leqq \rho \leqq 1$. *If* $\|T\| \leqq \dfrac{\rho}{2-\rho}$, *then* $T \in C_\rho$.

(ii) *For* $\rho \geqq 1$. *If* $\|T\| \leqq 1$, *then* $T \in C_\rho$.

Proof of Corollary 2. (ii) is clear and (i) is also derived from (i) of Theorem 4 replacing $\|T\|_N$ by $\|T\|$. q.e.d.

Theorem 5. *There exists* k *in* $[1/2, 1]$ *such that*

(i) *if* $T \in C_\rho$ *for* $0 \leqq \rho \leqq 1$, *then* $(2-\rho)\,\|T\|^2 k^2 + 2(1-\rho)\,\|T\|_N - \rho \leqq 0$.

(ii) *if* $T \in C_\rho$ *for* $1 \leqq \rho \leqq 2$, *then* $(2-\rho)\,\|T\|^2 k^2 + 2(\rho-1)\,\|T\|_N - \rho \leqq 0$.

Proof. Take sequences of unit vectors $\{h_n\}$ in (*) and (**) which $|(Th_n, h_n)|$ converges to $\|T\|_N$, then $\|T\|_N \leqq \sup \|Th_n\| \leqq \|T\|$. By this inequality and $1/2\|T\| \leqq \|T\|_N \leqq \|T\|$ ([4]), we get Theorem 5. q.e.d.

4. We consider an operator T which $\|T\|$ and $\|T\|_N$ equal s and $s/2$ respectively. For example $T_s = \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix}$. We can show $\|T_s\| = s$, $\|T_s\|_N = s/2$ and $r(T_s) = 0$ by simple calculations. Then by Theorem 4 we know

$$T \in \begin{cases} C_{\frac{2s^2+s}{s^2+s+1}} & \textit{if } 0 \leqq s \leqq 1 \\ C_{\frac{2s^2-s}{s^2-s+1}} & \textit{if } 1 \leqq s \leqq 2. \end{cases}$$

In [4] it is shown that $T_s \in C_{\frac{2s}{s+1}}$ if $0 \leqq s \leqq 1$. But by our estimation we get more precisely

$$T_s \in C_{\frac{2s^2+s}{s^2+s+1}} \subset C_{\frac{2s}{s+1}}.$$

However it is known by Durszt [2] that this operator belongs to more narrow class C_s. On the other hand we get the following inequality by Theorem 3

$$\|T_s\|_N = s/2 \leqq \begin{cases} \dfrac{s}{2-s} & \text{if } 0 \leqq s \leqq 1 \\ 1 & \text{if } 1 \leqq s \leqq 2. \end{cases}$$

Thus we know Theorem 3 and 4 give sharpenings of Theorem C exactly.

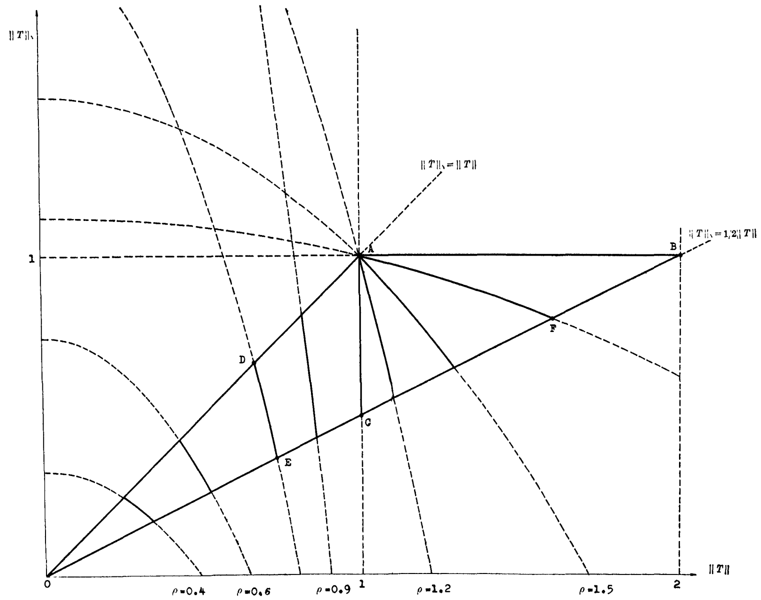
5. Theorem 4 indicates a sufficient condition for $T \in C_\rho$ $(0 \leqq \rho \leqq 2)$ depending upon $\|T\|$ and $\|T\|_N$ together. We can represent the relation among operator norm $\|T\|$, numerical radius $\|T\|_N$ and this sufficient condition by a domain ODE or OAF in a triangle OAB in the figure below. The curves DE and AF are given by

$$F_1(\rho) \equiv (2-\rho)\|T\|^2 + 2(1-\rho)\|T\|_N - \rho = 0 \qquad \textit{for } 0 \leqq \rho \leqq 1$$

$$F_2(\rho) \equiv (2-\rho)\|T\|^2 + 2(\rho-1)\|T\|_N - \rho = 0 \qquad \textit{for } 1 \leqq \rho \leqq 2$$

respectively.

When $\rho \to 1$, $F_1(\rho)$ and $F_2(\rho)$ gradually close to $\|T\|^2 - 1 = 0$ and the curves DE and AF close to the vertical line AC. Moreover $F_2(\rho)$ passes $A(1, 1)$ for every ρ and when $\rho \to 2$, $F_2(\rho)$ gradually close to $\|T\|_N - 1 = 0$ and the curve AF closes to the horizontal line AB. The triangular domains OAC and OAB indicate the necessary and sufficient condition for T to belong to C_1 and C_2 respectively. The line OA indicates the degenerated domain which give the necessary and sufficient condition for a normaloid operator T to belong to $C_\rho (0 \leqq \rho \leqq 1)$, where the coordinates of D are $\left(\frac{\rho}{2-\rho}, \frac{\rho}{2-\rho}\right)$ by Theorem 4 and Theorem D.



$(2-\rho)\|T\|^2+2(1-\rho)\|T\|_N-\rho\leqq 0$ for $0\leqq\rho\leqq 1$ $(2-\rho)\|T\|^2+2(\rho-1)\|T\|_N-\rho\leqq 0$ for $1\leqq\rho\leqq 2$

References

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