3. *An Extension of Beurling's Theorem. II*

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This article is the continuation of the previous paper of the same title. We shall prove the following

Lemma 5. *Let F be a set of positive capacity in B and suppose for any point $p\in F\cap B_1^N$ there exists a contact set $\varDelta(p)$ of p such that $\lim_{n=\infty}{}_{\varDelta(p)\cap v_n(p)}N(z,p)>0$. Let $\{G_n\}$ be a decreasing sequence of domains in $R-R_0$ such that $G_n\supset(G_{n,p}\cap v_m(p)\cap\varDelta(p))$, where $G_{n,p}\overset{N}{\ni}p$ and m and $G_{n,p}$ depend on p and G_n. Then*

$$\omega(\{G_n\},z)>0.$$

Put $\varGamma_m=E\left[p\in B_1^N:\int_{\partial R_0}\frac{\partial}{\partial n}\lim_{n=\infty}{}_{G_{n'},p\cap v_n(p)\cap\varDelta(p)}N(z,p)ds\geqq\frac{2\pi}{m}\right]$. Then by Lemma 1.3) $F=B_0^N+\sum_{m=1}^{\infty}\varGamma_m$. Now since B_0^N is an F_σ set of capacity zero, there exists a number l_0 and a closed set F' of positive capacity in F such $(F'\cap B_1^N)\subset\varGamma_{l_0}$. Hence there exists a positive mass distribution μ on $F'\cap B_1^N$ such that $V(z)=\int N(z,p)d\mu(p)$ and $V(z)\leqq1$ in $R-R_0$. ${}_{G_n}V(z)={}_{G_n}\left(\int N(z,p)d\mu(p)\right)\geqq\int\lim_{n=\infty}{}_{G_{n'},p\cap\varDelta(p)\cap v_n(p)}N(z,p)d\mu(p)$ $=\int\lim_{n=\infty}{}_{\varDelta(p)\cap v_n(p)}N(z,p)d\mu(p)$ for any n and $\int_{\partial R_0}\frac{\partial}{\partial n}{}_{G_n}V(z)ds\geqq\frac{2\pi}{l_0}\int d\mu(p)>0$. Let $\omega_n(z)=\omega(G_n,z,R-R_0)$. Then by the maximum principle $\omega_n(z)\geqq{}_{G_n}V(z)$. Let $n\to\infty$. Then

$$\omega(\{G_n\},z)\geqq\lim_{n=\infty}{}_{G_n}V(z)>0.$$

Let $w=f(z)$: $z\in R$ be an analytic function whose values fall on a basic surface $\underline{R}$. Suppose N-Martin's topology is defined in $\bar{R}-R_0$. Let $p\in B_1^N$ and let $\varDelta(p)$ be a contact set of $p\in B_1^N$. Put

$$M(f(p))=\bigcap_\tau\overline{f(G_\tau)}\ \text{and}\ \varDelta(f(p))=\bigcap_\tau\overline{f(\varDelta(p)\cap G_\tau)}.$$

Then $M(f(p))\subset\varDelta(f(p))$, where $\{G_\tau\}$ runs over all domains G_τ such that $G_\tau\overset{N}{\ni}p$ and the closure is taken with respect to the topology of $\underline{R}$.

Let $\mathfrak{F}$ be a closed set in R. We suppose $\mathfrak{F}$ is contained in a local parameter disc $|w|<1$ and let $A(r)$ be the area of R (not of R) on $E[w:\text{dist}\,(w,\mathfrak{F})\leqq r]=\mathfrak{F}_r$. We suppose $\mathfrak{F}$ in $|w|<\frac{1}{2}$. If $\mathfrak{F}$ is one point a and $\varliminf_{r\to0}\frac{A(r)}{r^2}<\infty$, a is called an ordinary point. A. Beurling and M. Tsuji proved the following

Theorem (B). (*A. Beurling*)[1]. *Let $w=f(z)$ be a non const. analytic function in $|z|<1$ and $w\in w$-Riemann sphere. Suppose the speherical area of the covering surface generated by $w=f(z)$ is finite. Let a be an ordinary point. Then the set F of $e^{i\theta}$ such that $\lim_{r\to 1} f(r\,e^{i\theta})=a$ is a set of capacity zero.*

Theorem (T). (*M. Tsuji*) [2]. *Let $w=f(z)$ be an analytic function in $|z|<1$: $w\in w$-Riemann sphere. Suppose there exist three values such that $f(z)$ takes these values only a finite number of times in $|z|<1$. Let F be a set of positive capacity on $|z|=1$ such that for any $e^{i\theta}\in F$ there exists a curve γ terminating at $e^{i\theta}$ and that $f(z)$ tends a as z tends $e^{i\theta}$ along γ. If a is an ordinary point, then $f(z)\equiv a$.*

We shall prove

Theorem (K). *Let $w=f(z)$ be a non const. analytic function from R into . $\underline{R}$ Let $\mathfrak{F}$ be a closed set contained in a local parameter disc $|w|<1$ and $\mathfrak{F}$ is contained in $|w|<\frac{1}{2}$. Suppose $\varlimsup_{r\to 0}\frac{A(r)}{r^2}<\infty$. Then $F=E[p\in B_1^N: \varDelta(f(p))\subset\mathfrak{F}]$ is a set of capacity zero.*

Remark. 1) [3]. In the previous paper we proved $F=E[p\in B_1^N$: $M(f(p))\subset\mathfrak{F}]$ *is a set of capacity zero* under the same condition of Theorem (K).

(2). By Lemma 3) a curve terminating at $e^{i\theta}$ is a contact set of $e^{i\theta}$ and B_0^N is a set of capacity zero. *Hence* (K) *implies* (T).

(3). The condition $\varlimsup_{r\to 0}\frac{A(r)}{r^2}<\infty$ is not a condition for the set $\mathfrak{F}$, for instance $\mathfrak{F}$ may be a set of positive areal measure. The condition only means the part of R over $\mathfrak{F}_r$ is small so that (K) is valid. We constructed [4] a covering surface R over the w-plane with finite area such that R has a singular point $p\in B_s^N$ with the following properties: 1) $\omega(p,z)>0$, $\lim_{n=\infty} f(v_n(p))=$ one point. This example shows that *even when $\mathfrak{F}$ is one point, some condition is necessary for the validity of* (K).

Proof of Theorem (K). Let $\{R_n\}$ be an exhaustion of R with compact relative boundary ∂R_n. We suppose N-Martin's topology is defined in $\bar{R}-R_0$. Let $g_i=E[w\colon \mathrm{dist}(w,\mathfrak{F})<r_i]$ and let $A(r_i)$ be the area of the image $f^{-1}({}_r\mathfrak{F}_i)$ of $\mathfrak{F}_{r_i}$. By $\varlimsup_{r\to 0}\frac{A(r)}{r^2}<\infty$, there exists a sequence $r_1>r_2, \cdots$ such that $\frac{A(r_i)}{r_i^2}\leqq K<\infty$: $i\geqq 0$. and $\lim_{i=\infty} r_i=0$. Also by $A(r_i)\to 0$ as $i\to\infty$, we can find a number i_0 and a compact dist $\varGamma_0$ in $R_0-f^{-1}(g_i)$. We can suppose loss of generality $\frac{A(r_i)}{r_i^2}\leqq K$ and

$R-f^{-1}(g_i)\supset \Gamma_0$ for $i\geqq 0$. Put $G_0=f^{-1}(g_0)$ and $G_i=f^{-1}(g_i)\cap(R-R_i)$ for $i\geqq 1$. By the definition of $\Delta(f(p))$, for any i there exists a domain $G(p)$ such that $G(p)\overset{N}{\ni} p$ and $(\Delta(p)\cap G(p))\subset f^{-1}(g_i)$ for any $p\in F$. Now $R-R_i\supset v_n(p)\overset{N}{\ni} p$ (where n depends on i). Put $G'(p)=(R-R_i)\cap G(p)$, then $G'(p)\overset{N}{\ni} p$ and $(G'(p)\cap \Delta(p)\cap v_n(p))\subset G_i$.

Assume F is of positive capacity. Then by Lemma 5 $\omega(\{G_n\}, z, R-R_0)>0$. By $\Gamma_0\subset R_0$ we have by the Dirichlet principle and maximum principle

$$\omega(\{G_i\}, z, R-\Gamma_0)\geqq\omega(\{G_i\}, z, R-R_0)>0 \text{ and } \infty>D(\omega\{G_i\}, z, R-R_0)) \geqq D(\omega(\{G_i\}, z, R-\Gamma_0))>0. \tag{1}$$

We shall show G_{i+j} and $CG_j (j\geqq 1)$ are Dirichlet-disjoint. In fact, dist $(\partial g_i, \partial g_{i+j})>0$, ∂g_i, and ∂g_{i+j} are compact in the w-local parameter disc. We can construct domains Ω_1 and Ω_2 such that $g_i\supset\Omega_1\supset\Omega_2\supset g_{i+j}$, dist $(\partial\Omega_1, \partial\Omega_2)>0$ and $\partial\Omega_i (i=1, 2)$ is a finite number of analytic curves. Hence we can define a C_1-function $V'(w)$ such that $V'(w)$ is harmonic in $\Omega_1-\Omega_2$, $V'(w)=1$ in Ω_2, $V'(w)=0$ in $C\Omega_1$ and $\left|\frac{\partial}{\partial u}V'(w)\right|\leqq L$, $\left|\frac{\partial}{\partial V}V'(w)\right|\leqq L$: $L<\infty$ and $w=u+iv$. Put $V'(z)=V'(f(w))$ in G_0. Then $V'(z)=0$ in $G_0\cap Cf^{-1}(g_i)$, $V'(z)=1$ in $f^{-1}(g_{i+j})$ and $D(V'(z))\leqq KL^2r_0^2<\infty$. Let $\omega(z)$ be a C_1-function in $R (i\geqq 1, j\geqq 1)$ such that $\omega(z)=0$ in R_i, $\omega(z)$ is harmonic in $R_{i+j}-R_i$ and $\omega(z)=1$ in $(R-R_{i+j})$. Then since $\partial R_i\cap\partial R_{i+j}=0$, $D(\omega(z))<\infty$. Put $V(z)=\min(V'(z), \omega(z))$. Then $V(z)$ is also a C_1-function in $G_0=f^{-1}(g_0)$ such that $V(z)=0$ G_0-G_i, $V(z)=1$ on $G_{i+j}(=(R-R_{i+j})\cap f^{-1}(g_{i+j}))$ and $D(V(z))\leqq 2(D(V'(z))+D(\omega(z)))\leqq L'<\infty$, because $\left|\frac{\partial V(z)}{\partial x}\right|\leqq\left|\frac{\partial\omega(z)}{\partial x}\right|+\left|\frac{\partial V'(z)}{\partial x}\right|$ and $\left|\frac{\partial V(z)}{\partial y}\right|\leqq\left|\frac{\partial\omega(z)}{\partial y}\right|+\left|\frac{\partial V'(z)}{\partial y}\right|$. Hence G_{i+j} and CG_i are Dirichlet-disjoint and clearly CG_0 and G_{i+j} are Dirichlet-disjoint. Hence by (1)

$$\omega(\{G_n\}, z, G_0)\leqq\omega(\{G_n\}, z, R-\Gamma_0) \text{ and } \infty>L'\geqq D(\omega(\{G_n\}, z, G_0)) \geqq D(\omega(\{G_n\}, z, R-\Gamma_0))>0.$$

The domain $G_0=f^{-1}(g_0)$ is non compact and consists of enumerably infinite number of components. Put $U(z)=\omega(\{G_n\}, z, G_0)$. Then $U(z)$ is harmonic in G_0. Let G' be a compact component of G_0. Then $U(z)$ is harmonic in G' and $U(z)=0$ on $\partial G'$ and $U(z)\equiv 0$ in G'. Hence by $\underset{G_0}{D}(U(z))>0$, there exists at least one component G of G_0 such that $\infty>\underset{G}{D}(U(z))>0$ and $U(z)$ is non const. in G. In the following we fix G and consider $U(z)$ in G. By $U(z)>0$ in G we see G contains not empty components of $f^{-1}(g_i)$ for $i\geqq 1$. Since CG_i and G_{i+j} are Dirichlet-disjoint we have by Lemma 4 $\int_{\partial V_M-G_i}\frac{\partial}{\partial n}U(z)ds\downarrow 0$ as $M\uparrow 1$

for any given G_i and $\int_{\partial V_M} \frac{\partial}{\partial n} U(z)ds = D(U(z)) = \alpha$ for a regular level curve ∂V_M:

$$V_M = E[z \in G: U(z) > M]. \tag{2}$$

Let $U_n^M(z)$ be a harmonic function in $(R_n \cap G) - V_M$ such that $U_n^M(z)=0$ on ∂G, $U_n^M(z)=M$ on $\partial V_M \cap R_n$, $\frac{\partial}{\partial n} U_n^M(z)=0$ on $(G \cap \partial R_n) - V_M$. Then $U_n^M(z) \Rightarrow U(z)$ in $CV_M \cap G$ as $n \to \infty$ and $D(U_n^M(z)) \to MD(U(z))$ as $n \to \infty$. Suppose ∂V_M is regular, then $\int_{\partial V_M} \frac{\partial}{\partial n} U(z)ds = D(U(z)) = \alpha$ and by $\varliminf_{n=\infty} \int_{\partial V_M \cap G_i \cap R_n} \frac{\partial}{\partial n} U_n^M(z)ds \geqq \int_{\partial V_M \cap G_i} \frac{\partial}{\partial n} U(z)ds$ we can find for any given $\varepsilon > 0$ and given number i a number $M(\varepsilon)$ and a number $n_1(\varepsilon, i, M(i)) > i$ such that

$$\int_{\partial V_M \cap R_n \cap G_i} \frac{\partial}{\partial n} U_n^M(z)ds \geqq \alpha - \varepsilon \text{ for } n \geqq n_1(\varepsilon, i, M). \tag{3}$$

Fix ε and M at present and put

$$D(f^{-1}(g_k)) = \iint_{f^{-1}(g_k) \cap G} |f'(z)| \left\{ \left(\frac{\partial}{\partial x} U(z)\right)^2 + \left(\frac{\partial}{\partial y} U(z)\right)^2 \right\}^{\frac{1}{2}} dxdy \quad \text{and}$$

$$D(f^{-1}(g_k), CV_M \cap R_n) = \iint_{f^{-1}(g_k) \cap G \cap CV_M \cap R_n} |f'(z)| \left\{ \left(\frac{\partial}{\partial x} U_n^M(z)\right)^2 + \left(\frac{\partial}{\partial y} U_n^M(z)\right)^2 \right\}^{\frac{1}{2}} dxdy.$$

Then these are independent to change of local parameters. By $\underset{G}{D}(U(z)) < \infty$ we can find a number n such that $D(U(z))$ over $R - R_n < \frac{\varepsilon}{2}$. Next since $f(z)$ is analytic in R, the area of $f^{-1}(g_k)$ in $R_n \to 0$ as $k \to \infty$. We can find a number $k(\varepsilon)$ such that $D(U(z))$ over $f^{-1}(g_k) \cap R_n < \frac{\varepsilon}{2}$. Hence we can find a number $k(\varepsilon)$ such that $D(U(z))$ over $f^{-1}(g_k) < \varepsilon$. Fix such $k(\varepsilon)$. On the other hand, by Schwarz's inequality

$D(f^{-1}(g_k))^2 \leqq \iint_{f^{-1}(g_k) \cap G} |f'(z)|^2 \, \mathrm{dxdy} \, D(U(z)) \leqq A(r_k)\varepsilon \leqq Kr_k^2 \varepsilon$ for $k \geqq k(\varepsilon)$ and $D(f^{-1}(g_k)) \leqq \sqrt{K\varepsilon}\, r_k$. Put

$$D(f^{-1}(g_k), CV_M) = \iint_{CV_M \cap f^{-1}(g_k) \cap G} |f'(z)| \left\{ \left(\frac{\partial}{\partial x} U(z)\right)^2 + \left(\frac{\partial}{\partial y} U(z)^2\right) \right\}^{\frac{1}{2}} dxdy.$$

Then by $U_n^M(z) \Rightarrow U(z)$, $D(f^{-1}(g_k), CV_M \cap R_n) \to D(f^{-1}(g_k), CV_M)$ as $n \to \infty$. Hence for the ε we can find a number $n_2(\varepsilon, i, M)$ such that $D(f^{-1}(g_k), CV_M \cap R_n) \leqq D(f^{-1}(g_k), CV_M) + \varepsilon \leqq D(f^{-1}(g_k)) + \varepsilon$ for $n \geqq n_2$. Hence by (3) for the given ε and $i > k$ we can find a number $n_3 = \max(n_1, n_2)$ such that both following inequalities hold

$$\int_{G_i\cap R_n\cap\partial V_M}\frac{\partial}{\partial n}U_n^M(z)ds>\alpha-\varepsilon \text{ and } D(f^{-1}(g_k), CV_M\cap R_n)\leqq\sqrt{K\varepsilon r_k}+\varepsilon \quad (4)$$

for $n\geqq n_3$.

We see easily the number of components of ∂g_i in the w-local parameter disc is finite, because $|w|\leqq\frac{2}{3}$ is compact. Since $f(z)$ is analytic in R, $f^{-1}(\partial g_k)$ does not cluster in R. The function $U(z)$ is harmonic in G and ∂V_M does not cluster in R. If we deform ∂R_n slightly, then $U_n^M(z)$ and $\frac{\partial}{\partial n}U_n^M(z)$ vary very little. Hence we can deform ∂R_n so that

$$\int_{G_i\cap R_n\cap\partial V_M}\frac{\partial}{\partial n}U_n^M(z)ds\geqq\alpha-2\varepsilon \text{ and } D(f^{-1}(g_k), CV_M\cap R_n)\leqq\sqrt{K\varepsilon}\,r_k+2\varepsilon \quad (4')$$

and that the number of components of the boundary $(R_n\cap G)-V_M$ may be finite, where M is a fixed number.

Hence there are only a finite number of branch points of level curves of $U_n^M(z)$ in $(R_n\cap G)-V_M$. Put $z=\exp(U_n^M(z)+iV_n^M(z))=re^{i\theta}$, where $V_n^M(z)$ is the conjugate harmonic function of $U_n^M(z)$. Then $|z|=e^M$ on $\partial V_M\cap R_n$ and by (4′) $\int_{\partial V_M\cap R_n\cap G_i}d\theta=\int_{\partial V_M\cap R_n\cap G_i}\frac{\partial}{\partial n}U_n^M(z)ds\geqq\alpha-2\varepsilon$.

We consider $D(f^{-1}(g_k), CV_M\cap R_n)$ in the following way

$$D(f^{-1}(g_k), CV_M\cap R_n)=\iint_{f^{-1}(g_k)\cap CV_M\cap R_n\cap G}|f'(z)|\left\{\left(\frac{\partial}{\partial r}U_n^M(z)\right)^2+\frac{1}{r^2}\left(\frac{\partial}{\partial\theta}U_n^M(z)\right)\right\}^{\frac{1}{2}}r\,dr\,d\theta$$

$$\geqq\iint_{f^{-1}(g_k)\cap CV_M\cap R_n\cap G}|f'(re^{i\theta})|\left|\frac{\partial}{\partial r}U_n^M(re^{i\theta})\right|r\,dr\,d\theta=\iint_{f^{-1}(g_k)\cap CV_M\cap R_n\cap G}|f'(re^{i\theta})|dr d\theta$$

by $\left|\frac{\partial}{\partial r}U_n^M(z)\right|r=1$. Since $\frac{\partial}{\partial n}U_n^M(z)=0$ on ∂R_n, $\theta=$const. along ∂R_n. Let θ be the set of θ such that we can trace trajectry T_θ along which $\theta=$const. and $|z|$ varies continuousely from $|z|=1$ ($U_n^M(z)=0$ on $\partial G\cap R_n$) to e^M($U_n^M(z)=M$ on $\partial V^M\cap R_n$) in $G\cap CV_M\cap R_n$. Then since the connectivity of $G\cap R_n\cap CV_M$ is finite, mes $\theta=\int_{G\cap R_n\cap\partial V_M}\frac{\partial}{\partial n}U_n^M(z)ds$.

Let θ be the set of $\theta\in\theta$ such that T_θ intersects ∂G_i when z goes from ∂G to ∂V_M along T_θ. Then by (4) mes $\Theta\geqq\alpha-2\varepsilon$. Now $G_i=((R-R_i)\cap f^{-1}(g_i))\subset f^{-1}(g_i)\subset G=$one component of $f^{-1}(g_0)$. Hence the image of $f^{-1}(T_\theta)$ of $T_\theta:\theta\in\Theta$ must intersect $f^{-1}(\partial g_i)$ at least once and intersect $f^{-1}(g_k)$ at least once for $k:1<k<i$ when $|z|$ varies from 1 to e^M along T_θ. Hence $\int_{T\theta\cap CV_M\cap f^{-1}(g_k)\cap G}|f'(re^{i\theta})|\,dr\geqq r_k-r_i$ for $\theta\in\Theta$. Hence by (4′)

$$\sqrt{\varepsilon K r_k}+2\varepsilon \geqq D(f^{-1}(g_k))+2\varepsilon \geqq D(f^{-1}(g_k), CV_M \cap R_n) \geqq (\alpha-2\varepsilon)(r_k-r_i). \quad (5)$$

Now i is arbitrary. Let $i \to \infty$ and then $\varepsilon \to 0$. Then (5) is a contradiction. Thus the set F is of capacity zero.

As a corollary of Theorem (K) we shall prove

Corollary. *Let $\mathfrak{F}$ be a closed set in the w-plane and contained in* $|w|<\frac{1}{2}$. *If* $\varliminf_{r\to 0} \frac{\text{mes}\,\mathfrak{F}_r}{r^2}<\infty$: $\mathfrak{F}_r=E[w: \text{dist}\,(w, \mathfrak{F}) \leqq r]$, $\mathfrak{F}$ *is a set of logarithmic capacity zero.*

Let R: $|w|<1$ and $R=E[z \notin \mathfrak{F}: |z|<1]$ and $w=f(z)=z$. Then $f(z)$ is analytic in R. Let $R_0=E\left[z: \left|z+\frac{1}{6}\right|<\frac{1}{6}\right]$. Clearly R is a Riemann surface with positive boundary and N-Martin's topology is defined in $R-R_0$. Let B^* be the boundary of R (with respect to N-Martin's topology) whose projection is contained in $\mathfrak{F}$. Let $p \in (B^* \cap B_1^N)$. Then $M(f(p)) \subset \mathfrak{F}$. Hence by Theorem ($K$) B^* is a set of capacity zero. This means that there exists no non const. positive bounded full-harmonic function in $R-R_0$ with positive mass only on B^*. Assume $\mathfrak{F}$ is of positive logarithmic cap. Then $\lim_{r\to 0} {}_rU(z)>0$ and $\infty > D({}_rU(z))>0$ for $r_0 \geqq r>0$, where ${}_rU(z)$ is a harmonic function in $E[z \notin (R_0+\mathfrak{F}_r): |z|<1]$ such that ${}_rU(z)=0$ on $|z|=1$ and on ∂R_0, ${}_rU(z)=1$ on $\partial\mathfrak{F}_r$. Let $\omega_r(z)$ be a harmonic function in $E[z \notin (R_0+\mathfrak{F}_r)$: $|z|<1]$ such that $\omega_r(z)=0$ on ∂R_0, $\frac{\partial}{\partial n} w_r(z)=0$ on $|z|=1$ and $\omega_r(z)=1$ on $\partial\mathfrak{F}_r$. Then $\omega_r(z) \geqq {}_rU(z)$ and $\omega_r(z) \Rightarrow \omega(z)$ as $r \to 0$ and $D(\omega(z)) \leqq D({}_rU(z))<\infty$. Then $\omega(z)$ is full-harmonic in $R-R_0$ and mass of $\omega(z)$ lies only on B^*. This contradicts that B^* is of cap. zero. Hence $\mathfrak{F}$ is a set of logarithmic cap. zero.

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