19. *Remarks on Bounded Sets in Linear Ranked Spaces*

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One of the authors defined the boundedness in linear ranked spaces ([2], and [3] p. 590).

Definition 1. A subset B in a linear ranked space is called *bounded* if, for any non-negative integer n, there is an integer $m(m\geq n)$ and a neighbourhood V of the origin and of rank m which absorbs B.

In the first half of this note, we shall study some of their properties, and in the latter half, examine the definition of bounded sets.

Throughout this note, "*linear ranked space*" will mean a linear space over the real or complex field, where are defined families $\mathfrak{B}_n(n=0,1,2,\cdots)$ of circled subsets satisfying the axioms (A), (B), (a), (b), (1), (2), and (3) in the note [2].

§ 1. Some properties. We shall set two problems.

(I) *Is the r-closure*[3)] *of any bounded set also bounded?*

(II) *Let A be an unbounded set. Can we choose a countable sequence of points of A having no bounded subsequence?*

In general, their answers are all negative. We shall show it and give some conditions which make them positive.

About problem I: Example 1. (*The counter of* (I)) Let E be the linear space of all bounded real valued functions on real line. (Addition and scalar multiplication are usual.) We define the sets

$$V(k,n)=\left\{\varphi(t)\in E \;\middle|\; |t|>k \Rightarrow |\varphi(t)|<\frac{1}{n}\right\}$$

$$k,n=0,1,2,\cdots;\ \frac{1}{0}=+\infty.$$

The families $\mathfrak{B}_n=\{V(k,n)\mid k=0,1,2,\cdots\}$ $(n=0,1,2,\cdots)$ possess the properties (A), (B), (a), (b), (1), (2), and (3) in the note [2], so E becomes a linear ranked space with indicator ω_0.

The set $B=V(1,1)$ is bounded since, for any non-negative integer n, $\frac{1}{n+1}B\subseteq V(1,n)$. The r-closure $\mathrm{cl}(B)$ of B consists of all $\varphi(t)$

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3) For any subset A of a ranked space, the set of all points, each of which is an r-limit point of a countable sequence of points of A, is called the r-closure of A and denoted by $\mathrm{cl}(A)$.

such that, for some positive integer k (depending on φ), if $|t|>k$ then $|\varphi(t)|\leq 1$. Hence $\mathrm{cl}(B)$ is not bounded.

Proposition 1. *If, in a linear ranked space, one of the following conditions* (1) *and* (2) *is satisfied, then the r-closure of any bounded set is also bounded.*

(1) *For any* V *of* $\mathfrak{B}_n(n\geq 1)$ *there is a* U *of* $\mathfrak{B}_{n-1}$ *such that* $\mathrm{cl}(V)\subseteq U$.

(2) *For any* V *of* $\mathfrak{B}=\bigcup_{n=0}^{\infty}\mathfrak{B}_n$ *there is a positive number* ρ *such that* $\mathrm{cl}(\rho V)\subseteq V$.

Proof. From (1): trivial.

From (2): Let B be a bounded set. Then, for any non-negative integer n, there is an integer $m(m\geq n)$ and a V of $\mathfrak{B}_m$ which absorbs B: i.e. there is a positive number λ such that $\lambda B\subseteq V$. Because

$$\rho\lambda\,\mathrm{cl}(B)=\mathrm{cl}(\rho\lambda B)\subseteq \mathrm{cl}(\rho V)\subseteq V,$$

$\mathrm{cl}(B)$ is also bounded.

In each of the examples in the note [2], the conditions (1) and (2) in the above proposition are both satisfied.

About problem II: Example 2. (*The counter of* (*II*)). Let Ω be the set of all ordinal numbers inferior to ω_1, the first uncountable ordinal number, and E be the linear space of all bounded real valued functions $\varphi(\xi)$ on Ω, unequal to zero for at most countably many ξ's. (Addition and scalar multiplication are both point-wise.) We define the sets

$$V(\alpha,0)=E,$$

$$V(\alpha,n)=\left\{\varphi(\xi)\in E\;\middle|\;\sup_{\xi}|\varphi(\xi)|<\frac{1}{n},\ \xi>\alpha\Rightarrow\varphi(\xi)=0\right\}$$

$$(0\leq\alpha<\omega_1,\ n=0,1,2,\cdots).$$

Then the families $\mathfrak{B}_n=\{V(\alpha,n)\mid 0\leq\alpha<\omega_1\}$ $(n=0,1,2,\cdots)$ possess the properties (A), (B), (a), (b), (1), (2), and (3) in the note [2], so E becomes a linear ranked space with indicator ω_0.

The set $A=\bigcup\{V(\alpha,1)\mid 0\leq\alpha<\omega_1\}$ is not bounded. But, because $V(0,1)\subseteq V(1,1)\subseteq\cdots\subseteq V(\alpha,1)\subseteq\cdots$, for any sequence $\{\varphi_n\}_{n=0,1,2,\ldots}$ of points of A, there is a $V(\alpha,1)$ such that

$$\varphi_n\in V(\alpha,1)\qquad(n=0,1,2,\cdots).$$

Hence, from the boundedness of $V(\alpha,1)$, $\{\varphi_n\}$ is bounded.

We shall say that a linear ranked space *has a countable monotone basis* when, for each n, there is a sequence $\{V_k^n\}_{k=0,1,2,\ldots}$ of elements of $\mathfrak{B}_n$ satisfying the following conditions:

(1) $V_k^n\subseteq V_{k+1}^n\ (k=0,1,2,\cdots)$;

(2) *for any* $m(m\geq n)$ *and for any* $V\in\mathfrak{B}_m$, *there is a* k *such that* $V\subseteq V_k^n$.

Obviously, each of the examples in the note [2] has a countable monotone basis.

Proposition 2. *In a linear ranked space having a countable monotone basis, from any unbounded set A, we can choose a sequence of points of A having no bounded subsequence.*

Proof. Since A is unbounded, there is an integer N such that any V in $\mathfrak{B}_N$ does not absorb A. Hence we can choose a sequence $\{x_n\}_{n=0,1,2,\dots}$ of points of A, such that

$$\frac{1}{n+1}x_n \notin V_n^N \qquad (n=0, 1, 2, \cdots).$$

Let $\{x_{n_k}\}_{k=0,1,2,\dots}(n_0<n_1<\dots<n_k<\dots)$ be any subsequence of $\{x_n\}$, V be any member of $\mathfrak{B}_M(M\geq N)$, and ρ be any positive number. From the properties (1) and (2) of countable monotone basis , there is an integer k such that $\frac{1}{n_k+1}<\rho$ and $V_{n_k}^N$ containing V.

Then we have $\frac{1}{n_k+1}x_{n_k}\notin V_{n_k}^N$, so $\rho x_{n_k}\notin V$, because $V_{n_k}^N$ is circled and $V\subseteq V_{n_k}^N$. Hence $\{x_{n_k}\}$ is unbounded.

§ 2. **On the definition.** In a linear topological space, a set B is bounded if and only if, for any point x of B, ρx converges to the origin uniformly on B when ρ tends to zero. Hence we may define the bounded sets in a linear ranked space as follows.

Definition 2. A subset B in a linear ranked space is called *bounded*, if there is a fundamental sequence $\{V_n\}_{n=0,1,2,\dots}$ of neighbourhoods with respect to the origin of which any member V_n absorbs B.

In this sence, any subset of bounded set is bounded. Any one point set is bounded. Any finite sum, any finite union and any scalar multiple of bounded sets are also bounded ([3] p. 590).

Any bounded set by Definition 2 is also bounded by Definition 1. In a linear ranked space satisfying the following condition $(*_1)$, which is equivalent to $(*_1)$ in the note [3], these two definitions are equivalent:

$(*_1)$ *for any two $U\in\mathfrak{B}_m$ and $V\in\mathfrak{B}_n$, there is an $l\geq \max(m, n)$ and a $W\in\mathfrak{B}_l$ which is included in $U\cap V$ and absorbs $U\cap V$.*

Hereafter, we shall use Definition 2.

We have the following proposition without any assumption ([3] p 591 Proposition 2).

Proposition 3. *For any bounded sequence $B=\{x_n\}_{n=0,1,2,\dots}$ of points and for any sequence $\{\varepsilon_n\}_{n=0,1,2,\dots}$ of numbers tending to zero, we have*

$$\{\lim \varepsilon_n x_n\}\ni 0.$$

Proof. Let $\{V_n\}_{n=0,1,2,\dots}$ be a fundamental sequence of neighbourhoods with respect to the origin, and $\{\rho_n\}_{n=0,1,2,\dots}$ be a sequence of positive numbers such that, for any n, we have

$$\rho_n B \subseteq V_n.$$

Without loss of generality, we may assume that

$$\rho_0 \geq \rho_1 \geq \ldots \geq \rho_n \geq \ldots.$$

For any non-negative integer n, there is an integer m such that $|\varepsilon_{m+p}| \leq \rho_n$ $(p=0, 1, 2, \ldots)$. Since $\rho_n x_{m+p} \in V_n$, we have $\varepsilon_{m+p} x_{m+p} \in V_n$ $(p=0, 1, 2, \cdots)$. Hence $\{\lim \varepsilon_n x_n\} \ni 0$.

From this proposition, the following proposition holds also without any assumption ([3] p. 592 Proposition 3).

Proposition 4. *Any continuous linear functional on a linear ranked space is bounded* ([3] p. 592 Definition 2).

When we use Definition 2, perhaps the convergent sequence is not always bounded. Of course, in a linear ranked space satisfying the condition $(*_1)$, any convergent sequence is bounded.

References

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