18. *Compactness in Ranked Spaces*

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It is the purpose of this note to study certain properties of sequentially compact sets in ranked spaces. Throughout this note, we shall always treat ranked spaces with indicator ω_0 ([2] p. 319), and $i, k, m, n, n_0, n_1, \ldots, n_k, \ldots$ will denote non-negative integers.

In a ranked space, for a point-sequence $\{x_n\}_{n=0,1,2,\ldots}$ and for a point x, if we have $x \in \{\lim_n x_n\}$ ([2] p. 319), then the sequence $\{x_n\}$ is said to *r-converge* to x, or the point x is said to be *an r-limit point* of $\{x_n\}$. The symbol $\mathcal{F}(x)$ will denote the collection of all fundamental sequences of neighbourhoods with respect to a point x ([3] p. 551).

Let A be a subset of a ranked space. If every countable sequence $\{x_n\}_{n=0,1,2,\ldots}$ of points of A contains a subsequence r-converging to a point of A, then A is said to be *r-compact*. The set of all points, each of which is an r-limit point of a countable sequence of points of A, is called the *r-closure* of A and denoted by $\mathrm{cl}(A)$. The set A is said to be *r-closed*, if we have $\mathrm{cl}(A)=A$.

We must take care about the r-convergence in a subset A of a ranked space E. The sequence of points of A, r-converging in the space E to a point x of A, r-converges also to x in the induced ranked space A ([3] p. 550), but the converse is not always true.[1)]

Example 1. The interval $I=[-2, 2]$ of real numbers with families $\mathfrak{V}_n(x)=\left\{\left(x-\frac{1}{n}, x+\frac{1}{n}\right)\cap I\right\}$ $(x \in I, n=0, 1, 2, \cdots)$[2)] becomes a ranked space with indicator ω_0 which will be denoted by E. $\left(\text{we put } \frac{1}{0}=+\infty.\right)$

For $x \in I$, let

$$\mathfrak{V}'_n(x)=\begin{cases}\mathfrak{V}_n(x) & \text{when } x\neq 0, \text{ or when } x=0,\ n=0.\\ \left\{\left(-\frac{1}{n}, \frac{1}{n}\right), \left(-2+\frac{1}{n}, 2-\frac{1}{n}\right)\right\} & \text{when } x=0,\ n>0.\end{cases}$$

Then I with $\mathfrak{V}'_n(x)$ $(x \in I, n=0, 1, 2, \cdots)$ also becomes another ranked space with indicator ω_0 which will be denoted by E'.

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1) A condition which makes the converse hold was given in [6] Proposition 15.

2) $\mathfrak{V}_n(x)$ will denote the family of neighbourhoods of point x and of rank n. See [5] p. 616.

In the induced space A' of E', consisting of all points x such that $|x|<1$, any point-sequence r-converges to 0. Therefore the sequence $\left\{(-1)^n \cdot \frac{1}{2}\right\}_{n=0,1,2,\dots}$ r-converges in A', though it does not in E'.

The r-convergence in the above definitions will mean that in the whole ranked space. Therefore, if a subset A of a ranked space E is r-compact, then the induced ranked space A of E is also r-compact.

§ 1. **Fundamental properties.** **Proposition 1.** *In a ranked space:*

(1) *any one point set is r-compact;*

(2) *any finite union of r-compact sets is r-compact;*

(3) *any r-closed subset of an r-compact set is r-compact.*

Proposition 2. *The image by an r-continuous mapping* ([3] p. 550) *of an r-compact set is r-compact.*

In Example 1, two induced subspaces $A=(-1,1)$ and $B=[1,2]$ of E' are both r-compact. But induced space $A\cup B$ of E' is not r-compact. The natural mapping $i: E'\to E$ is r-continuous. But the induced space $i(A)$ of E is not r-compact, though the induced space A of E' is r-compact.

A ranked space is said to be *r-separated* if it satisfies the following axiom:

(R_2) *for any two distinct point x and y, and for any members $\{U_n(x)\}$ of $\mathfrak{F}(x)$ and $\{V_n(y)\}$ of $\mathfrak{F}(y)$, there is an n such that*

$$U_n(x)\cap V_n(y)=\phi.$$

We can consider any metric space an r-separated ranked space. And each of the examples in the note [4] is r-separated.

In any r-separated ranked space, every point-sequence $\{x_n\}_{n=0,1,2,\dots}$ r-converges to at most one point. Hence the following holds.

Proposition 3. *Any r-compact subset of an r-separated ranked space is r-closed.*

A linear space which is at the same time a ranked space will be called a *linear ranked space* if addition and scalar multiplication are both r-continuous ([4]).

Proposition 4. *In a linear ranked space:*

(1) *any scalar multiple of an r-compact set is r-compact;*

(2) *any finite sum of r-compact sets is r-compact;*

(3) *the sum of an r-compact set and an r-closed set is r-closed.*

Proof. (1) and (2). These result from the r-continuity of addition and scalar multiplication.

(3) Let C be r-compact, F be r-closed and suppose that a sequence $\{x_n\}_{n=0,1,2,\dots}$ of points of $C+F$ r-converges to a point x. Any x_n can be represented by the form y_n+z_n where $y_n\in C$ and $z_n\in F$.

From the r-compactness of C, there is a subsequence $\{y_{n_i}\}_{i=0,1,2,\cdots}$ of $\{y_n\}$ r-converging to a point y of C. Because $\{x_{n_i}\}_{i=0,1,2,\cdots}$ also r-converges to x ([2] Proposition 1), and form the r-continuity of addition and scalar multiplication, $\{z_{n_i}=x_{n_i}-y_{n_i}\}$ r-converges to $x-y$. From the r-closedness of F, $x-y\in F$. Therefore x belongs to $C+F$, so $C+F$ is r-closed.

§ 2. **Coverings and r-compactness.** Let E be a ranked space and T be a set of indices. Suppose that, for any point x of E and for any index τ of T, there is a member of $\mathcal{F}(x)$ denoted by $\tau(x)$, and that any member of $\mathcal{F}(x)$ is inferior to a $\tau(x)$.[3]

Any ranked space may be considered to possess this property. In the case of metric spaces, T consists essentially one element. For any linear ranked space, in which every $\mathcal{F}(x)$ is obtained by translation of $\mathcal{F}(0)$ of the origin 0, we can identify T with a subcollection of $\mathcal{F}(0)$.

For any subset A of E, and for any τ of T, we define the sets

$$\mathrm{cl}_\tau(A)=\{x \mid x\in E, \forall V\in\tau(x)\ \ A\cap V\neq\phi\}.$$
$$\mathrm{in}_\tau(A)=\{x \mid x\in E, \exists V\in\tau(x)\ \ V\subseteq A\}.$$
$$\mathrm{in}(A)=\bigcap_\tau \mathrm{in}_\tau(A),$$

then we have

$$\mathrm{cl}(A)=\bigcup_\tau \mathrm{cl}_\tau(A),\qquad (\mathrm{cl}_\tau(A))^c=\mathrm{in}_\tau(A^c),\qquad (\mathrm{cl}(A))^c=\mathrm{in}(A^c),$$

where X^c denoted the compliment of subset X of E.

Theorem 1. *For any subset A of the ranked space E, the following three conditions are equivalent:*

(1) *A is r-compact;*

(2) *for any countable family $\{B_n\}$ of subsets of A, possessing the finite intersection property, there is a $\tau\in T$ such that we have $A\cap(\bigcap_n \mathrm{cl}_\tau(B_n))\neq\phi$.*

(3) *for any countable family $\{C_n\}$ of subsets of E such that, for every $\tau\in T$, $\{\mathrm{in}_\tau(C_n)\}$ covers A, there is a finite subfamily of $\{C_n\}$ covering A.*

Proof. (1)$\Rightarrow$(2). Let $\{B_n\}_{n=0,1,2,\cdots}$ be a family satisfying the supposition in (2) and, for any n, x_n be a point of $\bigcap_{k=0}^{n} B_k$. Then there is a point x of A which is an r-limit point of some subsequence of $\{x_n\}$. Therefore, for some τ of T, every $V_m(x)$ of $\tau(x)$ contains countably many terms of $\{x_n\}$, so we have

$$x\in \mathrm{cl}_\tau(B_n)\qquad (n=0,1,2,\cdots).$$

(2)$\Rightarrow$(3). Let $\{C_n\}_{n=0,1,2,\cdots}$ be a family of subsets of E, and suppose that every finite subfamily of $\{C_n\}$ does not cover A. Let us put

3) For two fundamental sequences $u=\{U_n(x)\}$ of and $v=\{V_n(x)\}$ of neighbourhoods with respect to a same point x, we say that v is inferior to u and write it $v<u$ when, for any $U_m(x)$ there is a $V_n(x)$ included in $U_m(x)$.

$B_n = A - \bigcup_{k=0}^{n} C_k (n=0, 1, 2, \cdots)$, then $\{B_n\}_{n=0,1,2,\ldots}$ is a countable family of subsets of A and possesses the finite intersection property. From (2), for some τ of T, there is a point x of A such that

$$x \in \bigcap_n \mathrm{cl}_\tau (B_n).$$

Because

$$\mathrm{in}_\tau (C_n) \subseteq \mathrm{in}_\tau (\bigcup_{k=0}^{n} C_k) \subseteq \mathrm{in}_\tau (B_n^c) = (\mathrm{cl}_\tau (B_n))^c,$$

we have

$$x \in \bigcup_n \mathrm{in}_\tau (C_n).$$

Hence $\{\mathrm{in}_\tau (C_n)\}$ does not cover A.

(3)$\Rightarrow$(1). Let $\{x_n\}_{n=0,1,2,\ldots}$ be a point-sequence of A. We may suppose that, if $m \neq n$, then $x_m \neq x_n$. Let us put

$$C_n = E - \{x_n, x_{n+1}, \cdots\} \qquad (n=0, 1, 2, \cdots)$$

then every finite subfamily of $\{C_n\}_{n=0,1,2,\ldots}$ does not cover A. Therefore, for some τ of T, there is a point x of A such that

$$x \notin \bigcup_n \mathrm{in}_\tau (C_n).$$

This means that $x \in \bigcap_n \mathrm{cl}_\tau (\{x_n, x_{n+1}, \cdots\})$. Let $\tau(x) = \{V_k(x)\}_{k=0,1,2,\ldots}$. We can choose a subsequence $\{x_{n_k}\}_{k=0,1,2,\ldots}$ from $\{x_n\}$ such that

$$n_0 < n_1 < n_2 < \cdots < n_k < \cdots,$$

$$x_{n_k} \in V_k(x) \qquad (n=0, 1, 2, \cdots),$$

so $\{x_n\}$ contains a subsequence r-converging to the point x of A. Hencee A is r-compact.

References

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