17. *A Condition of Convergent Filters*

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The object of this note is to give definitions of convergent filter which satisfy or don't satisfy a condition [1].

A family $\mathcal{F}$ of subsets of a set is said to be a *filter* if it possesses the following properties:

(1) The void set ϕ is not in $\mathcal{F}$,

(2) If $A \supseteq B$ and $B \in \mathcal{F}$, then $A \in \mathcal{F}$,

(3) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

If $\mathcal{F}$ and $\mathcal{F}'$ are filters of subsets of a set, we say that $\mathcal{F}$ is finer than $\mathcal{F}'$ if $\mathcal{F} \supseteq \mathcal{F}'$. A filter is called an *ultrafilter* if it is not fined by any filter but itself. The collection of all subsets of a topological space X which contain a neighborhood of a point p forms a filter $\mathfrak{N}(p)$. In general, a filter $\mathcal{F}$ of subsets of a topological space X converges to a point $p \in X$, if every neighborhood of p belong to $\mathcal{F}$. Thus the filter $\mathcal{F}$ converges to p if and only if $\mathcal{F}$ is finer than $\mathfrak{N}(p)$ (see [2]).

Lemma. *Let $x \in E$, then the collection $\dot{x}$ of all subsets of X which contain a point x is an ultrafilter on E.*

Definition. If a filter $\mathcal{F}$ is not convergent to x, then there exists $V \subseteq E$, $x \in V$, and $V \notin \mathcal{F}$ such that for each $y \in V$, $\mathfrak{G}$ converging to y includes V. When, we shall call that a set or a topological space E satisfies the condition (c).

Theorem 1. *Let E be a topological space, we define a filter $\mathcal{F}$ converge to x as $\mathcal{F} \supseteq \mathfrak{N}(x)$. i.e. $\mathcal{F} \to x \Longleftrightarrow \mathcal{F} \supseteq \mathfrak{N}(x)$, then the topological space E satisfies the condition (c). Moreover the inverse is true.*

Proof. Now we suppose $\mathcal{F} \not\to x$ and $\mathcal{F} \not\supseteq \mathfrak{N}(x)$, then there exists an open set $V \in \mathfrak{N}(x)$ in E, $V \notin \mathcal{F}$, and $x \in V$. Let $\mathfrak{G} \supseteq \mathfrak{N}(y)$ for each $y \in V$, of course, V is a neighborhood of y, therefore V is contained in $\mathfrak{G}$.

Conversely, we shall prove that if E satisfies the condition (c) then $\mathcal{F} \supseteq \mathfrak{N}(x)$ if and only if $\mathcal{F} \to x$. Let $\mathcal{F} \not\to x$, then there exists a set $V \subseteq E, x \in V$, $V \notin \mathcal{F}$ such that $V \in \mathfrak{N}(x)$. Hence $\mathcal{F} \not\supseteq \mathfrak{N}(x)$. On the other hand, let $\mathcal{F} \to x$, V be a open set in $\mathfrak{N}(x)$ then for each $y \in V$ and $\mathfrak{G} \to y$ we have $V \in \mathfrak{G}$. In particular, $x \in V, \mathcal{F} \to x$ so that $V \in \mathcal{F}$. Thus for each $V' \in \mathfrak{N}(x)$, there exists an open set V such that $V' \supseteq V$, $V \in \mathfrak{N}(x)$. By the above, the open set $V \in \mathcal{F}$ and $V' \in \mathcal{F}$ because $\mathcal{F}$ is a filter.

Theorem 2. *Let X, Y be topological spaces, and Φ a filter of* *subsets of all continuous function $\zeta(X, Y)$ from X into Y.*

We define continuous convergence:

$$\Phi \xrightarrow{c} \varphi \Longleftrightarrow \forall x \in X, \forall \mathcal{F} \to x \Rightarrow \Phi(\mathcal{F}) \to \varphi(x).$$

Where $\Phi(\mathcal{F})$ is the filter generated by $\{A(F) \mid A \in \Phi, F \in \mathcal{F}\}$ and $A(F) = \{f(t) \mid f \in A, t \in F\}$. Of course, $\Phi(\mathcal{F})$ is the filter on Y, the convergence of $\mathcal{F} \to x$ is in the topological space X, and the convergence of $\Phi(\mathcal{F}) \to \varphi(x)$ is in the topological space Y.

If we define continuous convergence in $\zeta(X, Y)$ then the topological space $\zeta(X, Y)$ don't satisfy the condition (c).

Proof. Let $\Phi \overset{c}{\not\to} \varphi$, then by definition of convergence there exists some filter $\mathcal{F}$ converging to x in X such that $\Phi(\mathcal{F}) \not\to \varphi(x)$. Hence $\mathcal{F} \supseteq \mathfrak{N}(x)$ and $\Phi(\mathcal{F}) \not\supseteq \mathfrak{N}(\varphi(x))$. Moreover there exists a neighborhood W of $\varphi(x)$ not containing in $\Phi(\mathcal{F})$. Each $W \in \mathfrak{N}(\varphi(x))$ not containing in $\Phi(\mathcal{F})$ forms $\{f(t) \mid f \in A', t \in F', \varphi \in A' \notin \Phi, x \in F' \mathcal{F}\}$. On the other we have $\varphi(\mathcal{F}) \supseteq \mathfrak{N}(\varphi(x))$ because $\mathcal{F} \supseteq \mathfrak{N}(x)$ and φ is continuous, $\varphi(\mathcal{F}) \supseteq \mathfrak{N}(\varphi(x))$ shows $\varphi(\mathcal{F}) \to \varphi(x)$ in Y. But the form of W is not contained in $\varphi(\mathcal{F})$. The fact mentioned above conflicts with the condition (c).

Theorem 3. *Let X be a set, Y a topological space.*

We define pointwise convergence:

$$\Phi \xrightarrow{p} \varphi \Longleftrightarrow \forall x \in X, \quad \Phi(\dot{x}) \to \varphi(x) .$$

Then the topological space $\zeta(X, Y)$ satisfies the condition (c).

Proof. Let $\Phi \overset{p}{\not\to} \varphi$, then there exists $x \in X$ such that $\Phi(\dot{x}) \not\to \varphi(x)$. The convergence of $\Phi(\dot{x}) \to \varphi(x)$ is in the topological space Y, by Theorem 1, we have a open neighborhood V of $\varphi(x)$ not containing in $\Phi(\dot{x})$. For each $y \in V$, a filter $\mathfrak{G} \to y$ satisfies $\mathfrak{G} \supseteq \mathfrak{N}(y)$ and $\mathfrak{N}(y) \ni V$. Hence $V \in \mathfrak{G}$.

Theorem 4. *Let X, Y be Banach spaces, B the unit sphere in X.*

We define bounded convergence:

$$\Phi \xrightarrow{b} 0 \Longleftrightarrow \Phi(B) \to 0$$

where Φ is a filter linear continuous functions $B(X, Y)$ from X into Y. Then the topological space $B(X, Y)$ satisfies the condition (c).

Proof. Let $\Phi \overset{b}{\not\to} 0$ then for each $\mathcal{F} \in \Phi$ there exists a positive number λ such that $\sup_{B, \mathcal{F}} \| \mathcal{F}(B) \| > \lambda$. Consider a set V consisting of $\{f \mid \sup_B \| f(B) \| < \lambda\}$, $V \notin \Phi$ and $0 \in V$. For each $f \in V$, there exists $\varepsilon > 0$ such that $\sup_B \| f(B) \| + \varepsilon < \lambda$. If a filter $\mathfrak{G}$ converges to f, i.e. $\mathfrak{G} - f = \{G - f \mid G \in \mathfrak{G}\} = \{\{g - f \mid g \in G\}\}$ then $\mathfrak{G} - f \to 0$ and in other words there exists $G \in \mathfrak{G}$ satisfying $\sup_{B\ G} \| (G - f)(B) \| < \varepsilon$, we have $\| g(B) \| \leqq \| (g - f)(B) \| + \| f(B) \| < \lambda$ for each $g \in G$. Hence G is included in V. $\mathfrak{G}$ is a filter and $\mathfrak{G}$ includes V.

References

[1] Clarence H. Cook: Topological Structures and Filters. University of Oklahoma Mathematics Service Committee (1966).
[2] Nelson Dunford and Jacob T. Schwartz: Linear Operators Part 1: General Theory. Interscience Publishers, Inc. New York (1958).