14. On Equivalences of Laws in Elementary Protothetics. II

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In our previous paper [1], we have proved the equivalences of the two laws (i.e., the law of development and the law on the limit of a function).

In this paper, we shall prove the equivalence of the theorems (a) and (a′) which have been called the *generalized law on the limit of a function*. The rules of inference, substitution and replacement used in the systems of elementary protothetics has in detail given in J. Słupecki [2], and our paper [1].

(a) $[f, q]\{[p]\{f(p)\} \equiv f(q)\cdot f(\sim(q))\}$,

(a′) $[f,r,s]\{[p,q]\{f(p,q)\} \equiv f(r,s)\cdot f(r, \sim(s))\cdot f(\sim(r), s)\cdot f(\sim(r), \sim(s))\}$.

To show the equivalence mentioned above, we shall first prove the following theorem.

Theorem 1. $[f,r,s]\{[f,q]\{[p]\{f(p)\} \supset f(q)\cdot f(\sim(q))\}\cdot [u,v]\{f(u,v)\} \supset f(r, s)\cdot f(r, \sim(s))\cdot f(\sim(r), s)\cdot f(\sim(r), \sim(s))\}$.

Proof. (1) $[f, q]\{[p]\{f(p)\} \supset f(q)\cdot f(\sim(q))\}$,

(2) $[u, v]\{f(u, v)\} \supset$

by replacing the variables u, v in the assumption (2) with a variables r, s, we obtain the following expression:

(3) $f(r, s)$. (2)

By a similar procedures, we obtain the following expression:

(4) $f(r, \sim(s))$, (2)

(5) $f(\sim(r), s)$, (2)

(6) $f(\sim(r), \sim(s))$, (2)

$$f(r, s)\cdot f(r, \sim(s))\cdot f(\sim(r), s)\cdot f(\sim(r), \sim(s)).$$

To obtain the consequent we have used the following theorem of the propositional calculus:

$$[p, q, r, s]\{p \supset (q \supset (r \supset (s \supset p\cdot q\cdot r\cdot s)))\},$$

therefore we complete the proof of Theorem 1.

Theorem 2. $[f, q]\{[f, r, s]\{[p, q]\{f(p, q)\} \supset f(r, s)\cdot f(r, \sim(s))\cdot f(\sim(r), s)\cdot f(\sim(r), \sim(s))\}\cdot [u]\{f(u)\} \supset f(q)\cdot f(\sim(q))\}$.

Proof. (1) $[f,r,s]\{[p,q]\{f(p,q)\} \supset f(r,s)\cdot f(r, \sim(s))\cdot f(\sim(r), s)\cdot f(\sim(r), \sim(s))\}$,

(2) $[u]\{f(u)\} \supset$

By replacing the variable u in the assumption (2) with a variable q, we obtain the following expression :

(3) $f(q)$, (2)

(4) $f(\sim(q))$, (2)

$f(q)\cdot f(\sim(q))$, (3; 4)

in the last line of the proof we have applied the following theorem of the propositional calculus:

$$[p, q]\{p\supset(q\supset p\cdot q)\}$$

consequently we complete the proof of Theorem 2. Therefore we have obtained Theorem 3 by applying the following theorem of the propositional calculus to Theorem 1 and Theorem 2:

$$[p,q,r,s]\{((p\supset q)\cdot r\supset s)\supset(((r\supset s)\cdot p\supset q)\supset((p\supset q)\equiv(r\supset s)))\}.$$

Theorem 3. $[f, q]\{[p]\{f(p)\}\supset f(q)\cdot f(\sim(q))\}\equiv[f, r, s]\{[p, q]\{f(p, q)\}\supset f(r, s)\cdot f(r, \sim(s))\cdot f(\sim(r), s)\cdot f(\sim(r), \sim(s))\}$.

Next we shall prove the following theorem:

Theorem 4. $[f, r, u, s]\{[f, q]\{f(q)\cdot f(\sim(q))\supset[p]\{f(p)\}\}\cdot f(r, s)$ $f(r, \sim(s))\supset\psi<f, u>(r)\}$.

Where, we introduce the following definition:

D1 $[f, p, q]\{\psi<f, p>(q)\equiv f(q, p)\}$.

Proof. (1) $[f, q]\{f(q)\cdot f(\sim(q))\supset[p]\{f(p)\}\}$,

(2) $f(r, s)$,

(3) $f(r, \sim(s))$

(4) $\chi<f, r>(s)$, (D2; 2)

where, we introduce the following definition:

D2 $[f, p, q]\{\chi<f, p>(q)\equiv f(p, q)\}$.

(5) $\chi<f, r>(\sim(s))$, (D2; 3)

(6) $\chi<f, r>(u)$, (1; 4; 5)

(7) $f(r, u)$, (D2, 6)

$\psi<f, u>(r)$:

therefore we complete the proof of Theorem 4. Further, we shall prove the following theorem:

Theorem 5. $[f, r, s, u, v]\{[f, q]\{f(q)\cdot f(\sim(q))\supset[p]\{f(p)\}\}\cdot f(r, s)\cdot f(r, \sim(s))\cdot f(\sim(r), s)\cdot f(\sim(r), \sim(s))\supset f(v, u)\}$.

Proof. (1) $[f, q]\{f(q)\cdot f(\sim(q))\supset[p]\{f(p)\}\}$,

(2) $f(r, s)$,

(3) $f(r, \sim(s))$,

(4) $f(\sim(r), s)$,

(5) $f(\sim(r), \sim(s))$

(6) $\psi<f, u>(r)$, (Theorem 4; 1; 2; 3)

(7) $\psi<f, u>(\sim(r))$, (Theorem 4; 1; 4; 5)

(8) $\psi<f, u>(v)$, (1; 6; 7)

$f(v, u)$, (D1; 8)

then we complete the proof of Theorem 5. Therefore we have obtained Theorem 6 by applying the following theorem of the propositional calculus to Theorem 5:

$$[p, q, r]\{(p \supset q)\cdot(p \supset r) \supset (p \supset q \cdot r)\},$$

Theorem 6. $[f, r, s]\{[f, q]\{f(q)\cdot f(\sim(q)) \supset [p]\{f(p)\}\}\cdot f(r, s)\cdot f(r, \sim(s))\cdot f(\sim(r), s)\cdot f(\sim(r), \sim(s)) \supset [u, v]\{f(u, v)\}\}$.

Next, we shall prove the following theorem.

Theorem 7. $[f, q, u]\{[f, r, s]\{f(r, s)\cdot f(r, \sim(s))\cdot f(\sim(r), s)\cdot f(\sim(r), \sim(s)) \supset [p, q]\{f(p, q)\}\}\cdot f(q)\cdot f(\sim(q)) \supset f(u)\}$.

Proof. (1) $[f, r, s]\{f(r, s)\cdot f(r, \sim(s))\cdot f(\sim(r), s)\cdot f(\sim(r), \sim(s)) \supset [p, q]\{f(p, q)\}\}$,

(2) $f(q)$,

(3) $f(\sim(q)) \supset$,

where, we introduce the next definitions:

D3 $[p]\{vr(p) \equiv (p \equiv p)\}$,

D4 $[f, p, q]\{\omega\Leftarrow f \Rightarrow(p, q) \equiv (f(p) \equiv vr(q))\}$.

By applying the theorem

$$[f, p, q]\{\omega\Leftarrow f \Rightarrow(p, q) \equiv f(p)\},$$

in elementary protothetics to the expression (2), we obtain the following expression:

(4) $\omega\Leftarrow f \Rightarrow(q, v)$, (D4; 2)

(5) $\omega\Leftarrow f \Rightarrow(q, \sim(v))$, (D4; 2)

(6) $\omega\Leftarrow f \Rightarrow(\sim(q), v)$, (D4; 3)

(7) $\omega\Leftarrow f \Rightarrow(\sim(q), \sim(v))$, (D4; 3)

(8) $\omega\Leftarrow f \Rightarrow(s, t)$, (1; 4; 5; 6; 7)

(9) $\omega\Leftarrow f \Rightarrow(u, u)$, (8)

$f(u)$. (D4; 9)

Therefore we complete the proof of Theorem 7. Then we easily obtain the following theorem from Theorem 7.

Theorem 8. $[f, q]\{[f, r, s]\{f(r, s)\cdot f(r, \sim(s))\cdot f(\sim(r), s)\cdot f(\sim(r), \sim(s)) \supset [p, q]\{f(p, q)\}\}\cdot f(q)\cdot f(\sim(q)) \supset [u]\{f(u)\}\}$. Therefore we have the following Theorem 9 by applying the following theorem of the propositional calculus to Theorem 6 and Theorem 8.

Theorem 9. $[f, q]\{f(q)\cdot f(\sim(q)) \supset [p]\{f(p)\}\} \equiv [f, r, s]\{f(r, s)\cdot f(r, \sim(s))\cdot f(\sim(r), s)\cdot f(\sim(r), \sim(s)) \supset [u, v]\{f(u, v)\}\}$.

Then we have Theorem 10 from Theorem 3 and Theorem 7.

Theorem 10. $[f, q]\{[p]\{f(p)\} \equiv f(q)\cdot f(\sim(q))\} \equiv [f, r, s]\{[p, q]\{f(p, q)\} \equiv f(r, s)\cdot f(r, \sim(s))\cdot f(\sim(r), s)\cdot f(\sim(r), \sim(s))\}$.

Theorem 10 states that the generalized law on the limit of a function of one argument is equivalent to that of a function of two arguments. Therefore we complete the proof of the equivalence of the generalized law on the limit of a function in elementary protothetics.

References

[1] K. Chikawa: On equivalences of laws in elementary protothetics. I. Proc. Japan Acad., **43**, 743–747 (1967).
[2] J. Słupecki: St. Leśniewski's protothetics. Studia Logica, **1**, 44–112 (1953).