28. *On Automorphisms of an Injective Module*

By Yasutaka SUZUKI

Mathematical Institute, Tôhoku University, Sendai

(Comm. by Kenjiro SHODA, M. J. A., March 12, 1968)

1. Statement of the main result. Throughout this paper we assume that every ring has an identity element and an R-module means a unital left R-module. Let $B=\mathrm{Hom}_R(M, M)$ be an R-endomorphism ring of an R-module M as a right operator domain of M. In this paper we shall be concerned with the following condition:

Condition (0). $Me \approx M$, $e=e^2 \in B$, implies $e=1$.

It is easy to see that if any isomorphism between two R-submodules of M can be extended to an automorphism of M, then M satisfies Condition (0). Our aim is to prove the following theorem.

Theorem 1. *Let M be an injective R-module with Condition (0). Then any isomorphism between two R-submodules of M can be extended to an automorphism of M.*

2. Left self-injective, regular rings with Condition (0). We denote the injective envelope [1] of an R-module A by $E(A)$. We write $N'{}' \supset N$ if N' is an essential extension of N. If X is a subset of a ring S, we define the left (resp. right) annihilator

$$l(X)=\{s \in S \mid sX=0\}$$

(resp. $r(X)$, similarly). We shall list a series of lemmas.

Lemma 2. *Let S be a left self-injective, regular ring. Then every left annihilator ideal A is generated by an idempotent.*

Proof. By the regularity of S, we have $r(A)=\bigcup_{e=e^2 \in r(A)} eS$. Then

$$A=l(r(A))=l(\bigcup_{e \in r(A)} eS)=\bigcap_{e \in r(A)} l(eS)=\bigcap_{S(1-e)\supset A} S(1-e).$$

But, for each $S(1-e)\supset A$, $E(A)'\supset S(1-e)\cap E(A)'\supset A$ and hence $E(A)=S(1-e)\cap E(A)\subset S(1-e)$ by the injectivity of $S(1-e)\cap E(A)$. Therefore $A=E(A)=Sf$ for some $f=f^2 \in S$.

Lemma 3. (J. von Neumann [7, Lemma 18]). *Let S be a regular ring. Then a principal left ideal of S is a two-sided ideal if and only if it is generated by a central idempotent.*

Lemma 4. (B. Eckmann and A. Schopf [1, 4.3]). *Let $v: A \to A'$ be an R-isomorphism, then v can be extended to an R-isomorphism of $E(A)$ onto $E(A')$.*

Lemma 5. *For any two idempotents e, f of a regular ring S, the following conditions are equivalent:*

(1) $eSf \neq 0$.

(2) $Se' \approx Sf'$ *for some* $0 \neq Se' \subset Se$ *and* $Sf' \subset Sf$.

Proof. (1) implies (2). There exists a non zero map $v\colon Se\to Sf$ since

$$\mathrm{Hom}_S(Se, Sf)\approx eSf\neq 0.$$

Since $\mathrm{Im}(v)$ is projective by the regularity of S,

$$Se\to \mathrm{Im}(v)\to 0$$

splits. From this, (2) follows immediately.

(2) implies (1). Let $Se'\approx Sf'$ for some $0\neq Se'\subset Se$ and $Sf'\subset Sf$. We may assume that e' is an idempotent by the regularity of S. Then we have easily a non zero map $Se\to Sf'$ since Se' is a direct summand of Se. Thus

$$eSf\approx \mathrm{Hom}_S(Se, Sf)\neq 0.$$

This completes the proof.

The following lemma is very interesting and useful.

Lemma 6. *Let S be a let self-injective, regular ring and e, f be idempotents with $eSf=0$. Then there exist central, orthogonal idempotents e', f' such that $Se\subset Se'$ and $Sf\subset Sf'$.*

Proof. By Lemma 2 and Lemma 3, $l(r(eS))$ and $r(l(Sf))$ are generated by central idempotents e' and f' respectively. And clearly $Se\subset Se'$ and $Sf\subset Sf'$. $eSf=0$ implies that e' and f' are orthogonal.

Proposition 7. *Let S be a left self-injective, regular ring with Condition (0) for the left regular module ${}_SS$. Then any isomorphism between two left ideals of S can be extended to an automorphism of the left S-module S.*

Proof. By Zorn's lemma there is a maximal isomorphism v between two left ideals X and Y which extends the given isomorphism. By the injectivity of S, the maximality of v and Lemma 4, there are idempotents e, f such that

$$X=S(1-e)\approx Y=S(1-f)$$

and that Se and Sf do not contain any mutually isomorphic left ideals. Then $eSf=0$ by Lemma 5 and hence there are central, orthogonal idempotents e', f' such that $Se\subset Se'$ and $Sf\subset Sf'$ by Lemma 6.

$$e'f=e'(f'f)=(e'f')f=0$$

implies $S(1-f)\supset Se'$. Since $S(1-e)\approx S(1-f)$,

$$S(1-e)\supset Sg\approx Se' \text{ for some } g=g^2.$$

Now Se' is an ideal, hence $Sg\subset Se'$. Since ${}_SS$ satisfies Condition (0).

$$S=S(1-e')\oplus Se'\approx S(1-e')\oplus Sg$$

implies $Sg=Se'$. Furthermore

$$(1-e')e=e-e'e=e-e=0$$

implies $S(1-e)\supset S(1-e')$. Then

$$X=S(1-e)\supset Se'\oplus S(1-e')=S.$$

Hence $X=Y=S$, completing the proof.

Corollary 8. *Let S be a left self-injective, regular ring with*

Condition (0) *for* ${}_SS$ *and* e, f *be idempotents. Then* $Se \approx Sf$ *if and only if* $S(1-e) \approx S(1-f)$.

3. Proof of Theorem 1. We denote by $\bar{A}$ the image of a subset A of a ring S under the canonical mapping of S onto $S/J(S)$, where $J(S)$ denotes the Jacobson radical of S.

Lemma 9. *Let* e, f *be idempotents of* $B = \mathrm{Hom}_R(M, M)$. *Then the following conditions are equivalent*:

(1) $Me \approx Mf$.

(2) $Be \approx Bf$.

(3) $\bar{B}\bar{e} \approx \bar{B}\bar{f}$.

Proof. The equivalence of (2) and (3) is found in (N. Jacobson [5, III, 8, Proposition 1]).

Now consider the following statements:

(1) $Me \approx Mf$.

(1′) There exist R-homomorphisms

$$x: Me \longrightarrow Mf \quad \text{and} \quad y: Mf \longrightarrow Me$$

with $exy = e, fyx = f$.

(2′) There exist x' and $y' \in B$ with $x' = ex'f, y' = fy'e$, and $x'y' = e, y'x' = f$.

(2) $Be \approx Bf$.

Then (1), (1′), (2′), and (2) are equivalent since x and y in (1′) induce x' and y' in (2′) respectively and conversely. This completes the proof.

Lemma 10. *Let* M *be an injective* R*-module. Then*

(1) $\bar{B}$ *is a left self-injective, regular ring.*

(2) *If* M *satisfies Condition* (0), *then so does* ${}_{\bar{B}}\bar{B}$.

Proof. (1) is proved in (G. Renault [8, Théorème 2.1]). If M is injective, then idempotents of $\bar{B}$ can be lifted modulo $J(B)$ [2, Theorem 4.1]. From this fact together with Lemma 9, (2) follows.

Proof of Theorem 1. Let M be an injective R-module with Condition (0). For any isomorphism between two R-submodules X, Y of M, there exists an extended isomorphism

$$E(X) = Me \approx E(Y) = Mf \text{ for some } e = e^2, f = f^2 \in B$$

by Lemma 4. Hence $\bar{B}\bar{e} \approx \bar{B}\bar{f}$ by Lemma 9. Since $\bar{B}$ is a left self-injective, regular ring with Condition (0) for ${}_{\bar{B}}\bar{B}$ by Lemma 10, we have $\bar{B}(\bar{1} - \bar{e}) \approx \bar{B}(\bar{1} - \bar{f})$ by Corollary 8. This implies

$$M(1-e) \approx M(1-f)$$

by using again Lemma 9, completing the proof.

Similarly we can also prove Theorem 1 in case M is a quasi-injective R-module ([2] and [6]).

Corollary 11. *Let* M *be a quasi-injective* R*-module with*

Condition (0). *Then any isomorphism between two R-submodules of M can be extended to an automorphism of M.*

4. Remarks on Condition (0). In this section we shall examine the properties of Condition (0) and of the following condition:

Condition (0′). $xy=1$ in a ring S implies $yx=1$.

Now consider the following statements:

(1) *M satisfies Condition* (0).

(2) *${}_B B$ satisfies Condition* (0).

(2′) *B satisfies Condition* (0′).

(3) *${}_{\bar{B}}\bar{B}$ satisfies Condition* (0).

(3′) *$\bar{B}$ satisfies Condition* (0′).

Then we have the following implications:

$$\begin{array}{ccc} (3) & \Rightarrow & (2) \Longleftrightarrow (1) \\ \Uparrow & & \Uparrow \\ (3') & \Longleftrightarrow & (2') \end{array}$$

Proof. (3′) implies (2′). Let $xy=1$ in B. Then $(yx)^2=y(xy)x=yx$ and $\bar{x}\bar{y}=\bar{1}$ implies $\bar{y}\bar{x}=\bar{1}$ by (3′). Hence $yx=1$.

(2′) implies (3′). Let $\bar{x}\bar{y}=\bar{1}$ in $\bar{B}$. Then xy is a unit, and there exists an inverse element z of xy. $xyz=1$ implies $yzx=1$ by (2′) and clearly $\bar{z}=\bar{1}$. Hence $\bar{y}\bar{x}=\bar{1}$.

Other implications are trivial.

Moreover, if M is injective, then we can easily see the equivalence of (1), (2), (2′), (3), and (3′).

N. Jacobson [4, Theorem 1] shows that if a ring S has the ascending or the descending chain condition for principal left ideals generated by idempotents, then S satisfies Condition (0′). Therefore quasi-Frobenius rings, for example, satisfy Condition (0′).

Corollary (Y. Utumi [9, Theorem 5.6]). *Let S be a left self-injective ring with Condition* (0′). *Then any isomorphism between two left ideals of S can be extended to an automorphism of the left S-module S.*

Corollary (M. Ikeda [3, Theorem 2]). *Let Q be a quasi-Frobenius ring. Then any isomorphism between two left* (*resp. right*) *ideals of Q can be extended to an automorphism of the left* (*resp. right*) *Q-module Q.*

References

[1] B. Eckmann and A. Schopf: Über injektive Moduln. Arch. Math., **4**, 75–78 (1953).

[2] C. Faith and Y. Utumi: Quasi-injective modules and their endomorphism rings. Arch. Math., **15**, 166–174 (1964).

[3] M. Ikeda: A characterization of quasi-Frobenius rings. Osaka Math. J., **4**, 203-209 (1952).
[4] N. Jacobson: Some remarks on one-sided inverses. Proc. Amer. Math. Soc., **1**, 352-355 (1950).
[5] ——: Structure of rings. Amer. Math. Soc. Colloquium Publications, **37** (1956).
[6] R. E. Johnson and E. T. Wong: Quasi-injective modules and irreducible rings. J. London Math. Soc., **36**, 260-268 (1961).
[7] J. von Neumann: On regular rings. Proc. Nat. Acad. Sci., **22**, 707-713 (1936).
[8] G. Renault: Anneau associé à un module injectif. C. R. Acad. Sc. Paris, **264**, 1163-1164 (1967).
[9] Y. Utumi: On continuous rings and self injective rings. Trans. Amer. Math. Soc., **118**, 158-173 (1965).