25. *Cohomology Operations in Iterated Loop Spaces*

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1. Introduction. In [3], Dyer and Lashof have determined the mod p homology structure of iterated loop spaces by use of extended p-th power operations, where p always denotes a prime. This operation is a generalization of H-squaring (for $p=2$) defined by Araki-Kudo [2], and operates on mod p homology group of H_p^∞-spaces X, especially iterated loop spaces. Let $Q_i^{(p)}: H_n(X; Z_p) \to H_{np+i}(X; Z_p)$ be Dyer-Lashof's extended powers. For odd p, we denote operations $Q_{(p)}^j: H_n(X; Z_p) \to H_{n+2j(p-1)}(X; Z_p)$, $j=0, 1, \cdots$, by $Q_{(p)}^j x = (-1)^{i+m(n^2+n)/2}(m!)^n Q_{(2j-n)(p-1)}^{(p)} x$, $x \in H_n(X; Z_p)$, $m=(p-1)/2$, and for $p=2$, $Q_{(2)}^j: H_n(X; Z_p) \to H_{n+j}(X; Z_p)$ by $Q_{(2)}^j x = Q_{j-n}^{(2)} x$.

The operation $Q_{(p)}^j$ has the following properties: 1. *$Q_{(p)}^j$ is a homomorphism*; 2. *For odd p, $Q_{(p)}^j x=0$ if* deg $x>2j$ *and* $Q_{(p)}^j x = x^p$ *if* deg $x=2j$, *and for $p=2$, $Q_{(2)}^j x=0$ if* $\deg x>j$ *and* $Q_{(2)}^j x = x^2$ *if* $\deg x=j$; 3. $Q_{(p)}^j(x\cdot y)=\sum_{k+l=j} Q_{(p)}^k x\cdot Q_{(p)}^l y$; 4. *$Q_{(p)}^j$ commutes with the suspension homomorphism σ associated with the fibering of the contractible total space, $\sigma Q_{(p)}^j = Q_{(p)}^j \sigma$.*

Our purpose is to determine the relation between $Q_{(p)}^j$ and the Steenrod reduced power operations ρ^n (squaring operations Sq^n for $p=2$). To state the results, we denote by ρ_*^n the dual operation of ρ^n, i.e., defined by

$$\langle \rho_*^n x, y\rangle = \langle x, \rho^n y\rangle \text{ for } x \in H_*(X; Z_p),\ y \in H^*(X; Z_p).$$

Let $\binom{a}{b}$ be the binomial coefficient with the following conversions: $\binom{a}{b}=0$ for a or $b<0$ and $\binom{a}{b}=1$ for $b=0, a\geq 0$. Δ denotes the homology Bockstein operation. Then we have

Main theorem. *For odd p,*

$$\rho_*^n Q_{(p)}^{n+s} = \sum_i (-1)^{n+i}\binom{s(p-1)}{n-pi} Q_{(p)}^{s+i}\rho_*^i,$$

$$\rho_*^n \Delta Q_{(p)}^{n+s} = \sum_i (-1)^{n+i}\binom{s(p-1)-1}{n-pi} \Delta Q_{(p)}^{s+i}\rho_*^i + \sum_i (-1)^{n+i+1}\binom{s(p-1)-1}{n-pi-1} Q_{(p)}^{s+i}\rho_*^i \Delta,$$

and for $p=2$

$$Sq_*^n Q_{(2)}^{n+s} = \sum_i \binom{s}{n-2i} Q_{(2)}^{s+i} Sq_*^i.$$

Applying the results of [3], the reduced power operations in $Q(K)=\Omega^\infty S^\infty K$ and $\Omega^k S^{k+l}$, $e>0$, are computable by the theorem.

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2. **Preliminaries.** Let Σ_p denotes the symmetric group of p letters and π be the cyclic subgroup of order p generated by a cyclic permutation T. $J^n\sum_p$ denotes the n-th join of Σ_p with itself. Briefly, an H_p^∞-space is an associative H-space with a map Θ_p^∞: $J^\infty\Sigma_p\times X^p\to X$ such that (1) Σ_p-equivariant, (2) normalized, where $X^p=X\times\cdots\times X$, p-times (see [3]). Let W be the usual acyclic π-free complex with a single π-generator e_i for each dimension i. (W is realized geometrically by the "infinite dimensional sphere"). $J^\infty\pi$ is naturally included in $J^\infty\Sigma_p$ and $C_*(J^\infty\pi)$ is also an acyclic π-free chain complex. So, we can identify $J^\infty\pi$ with W, and if X is an H_p^∞-space then Θ_p^∞ induces a homomorphism (see [3])

$$(\Theta_\pi)_*\colon H_*(W\times_\pi X^p; Z_p)\longrightarrow H_*(X; Z_p).$$

Theorem 1. (Dyer-Lashof). *Let $x_1, x_2, \cdots$ be a Z_p-basis of homogeneous elements, finite for each dimenion, of $H_*(X; Z_p)$. Then the homology classes represented by the following cycles form a Z_p-basis of $H_*(W\times_\pi X^p; Z_p)$;*

$$e_i\otimes_\pi x_j^p,\ j=1, 2, \cdots, i\geq 0,\ x_j^p=x_j\otimes\cdots\otimes x_j\ (p\text{-times}),$$

$$e_0\otimes_\pi(x_{j_1}\otimes\cdots\otimes x_{j_p}),\ j_s\neq j_t \text{ for some } s, t,$$

where $(j_1, \cdots, j_p)$ runs through each representative of the classes obtained by cyclic permutations of the indices.

Now, let X be an H_p^∞-space and $x\in H_*(X; Z_p)$. Then the extended power operation $Q_i^{(p)}$ is defined by

$$Q_i^{(p)}(x)=(\Theta_\pi)_*(e_i\otimes_\pi x^p).$$

By this definition the proof of the main theorem can be reduced to the computation of ρ_*^n-operations in $H_*(W\times_\pi X^p; Z_p)$. Next, we sketch the definition of the Streenrod reduced powers (for details, see [4]). As is seen in [4] we may assume that X is a finite regular cell complex. Let $u\in H_q(X; Z_p)$, and let $P(u)$ be the external reduced p power [4]. π operates on X trivially, and on $W\times X$ and on $W\times X^p$ by a diagonal action, then the diagonal map d: $W\times X \to W\times X^p$ is π-equivariant and induces a map d: $W\times X\to W\times_\pi X^p$. The projection: $W\times_\pi X^p\to W/\pi$ makes $H^*(W\times_\pi X^p; Z_p)$ an $H^*(W/\pi; Z_p)$-module. Similarly, $H^*(W\times_\pi X; Z_p)=H^*(W/\pi\times X; Z_p)$ is also an $H^*(W/\pi; Z_p)$-module, and d^* is an $H^*(W/\pi; Z_p)$-homomorphism. Let w_i be the generator of $H^i(W/\pi; Z_p)$, dual to the homology class represented by e_i. β denotes the cohomology Bockstein operation. Then writing $\nu(q)=(m!)^{-q}(-1)^{m(q^2+q)/2}$, we can define the Steenrod reduced powers for $u\in H^q(X; Z_p)$ by, for $p>2$,

$$\nu(q)d^*P(u)=\sum_i(-1)^i w_{(q-2i)(p-1)}\times\rho^i u+\sum_i(-1)^i w_{(q-2i)(p-1)-1}\times\beta\rho^i u$$

and for $p=2$, $\quad d^*P(u)=\sum_i w_{q-i}\times Sq^i u$.

Theorem 2. *Let $u_1, u_2, \cdots$ be a Z_p-basis of $H^*(X; Z_p)$ dual to $x_1, x_2, \cdots$ of theorem 1. Then we can choose elements $z_1, z_2, \cdots$ in $H^*(W\times_\pi X^p; Z_p)$, which are a Z_p-basis of ker d^*, such that $w_i\times Pu_j$, $i\geq 0, j\geq 1$, and $z_k, k\geq 1$ form a Z_p-basis of $H^*(W\times_\pi X^p; Z_p)$ and $w_i\times Pu_j$ is dual to $e_i\otimes_\pi x_j^p$ for all i, j.*

Proof of Theorem 1, 2. It is known [3], [4] that there are isomorphisms $H_*(W\times_\pi X^p; Z_p)\cong H_*(W\otimes_\pi H_*(X^p; Z_p))$, $H^*(W\times_\pi X^p; Z_p)\cong H^*(\mathrm{Hom}_\pi(W\otimes H_*(X^p; Z_p), Z_p))$. The basis $x_1, x_2, \cdots$ gives a direct sum splitting of $H_*(X; Z_p)$, i.e.,

$$H_*(X; Z_p)=\sum_j A_j,\ A_j\cong Z_p\{x_j\} \text{ and } H^*(X; Z_p)=\sum_j A_j^*,\ A_j^*\cong Z_p\{u_j\}.$$

So, we have the following decomposition as π-modules

$$H_*(X^p; Z_p)=\sum_j A_j^p+\sum A_{j_1}\otimes \cdots \otimes A_{j_p},$$

where $A_j^p=A_j\otimes\cdots\otimes A_j$ and the second summation runs over $j_1, \cdots, j_p$ with $j_s\neq j_t$ for some s, t. It is easily checked that π operates trivially on the first term and freely on the second term, and there is a Z_p-module B such that the second term is isomorphic to $Z_p(\pi)\otimes B$, where $Z_p(\pi)$ denotes the groupring of π over Z_p. Therefore we have

$$H_*(W\times_\pi X^p; Z_p)\cong\sum_j H_*(W\otimes_\pi A_j^p)+H_*(W\otimes_\pi Z_p(\pi)\otimes B),$$

$$H^*(W\times_\pi X^p; Z_p)\cong\sum_j H^*(W\otimes_\pi A_j^p)+H^*(W\otimes_\pi Z_p(\pi)\otimes B).$$

$H_i(W\otimes_\pi A_j^p)$ is generated by $e_i\otimes_\pi x_j^p$. Since $H_*(W\otimes_\pi Z_p(\pi))\cong Z_p$, $H_*(W\otimes_\pi Z_p(\pi)\otimes B)\cong B$. This proves Theorem 1. Next consider the cohomology group. It is proved in Chapter VIII of [4] that $H^i(W\otimes_\pi A_j^p)\cong H^i(W/\pi; Z_p)\otimes(A_j^*)^p$ is generated by $w_i\times Pu_j$ and that $H^*(W\otimes_\pi Z_p(\pi)\otimes B)\subset\ker d^*$. Now we shall prove that $H^*(W\otimes_\pi Z_p(\pi)\otimes B)=\ker d^*$. Let $z=\sum_k a_k w_{i_k}\times Pu_{j_k}$, $a_k\ Z_p$, be a homogeneous element such that $d^*z=0$. That is, for odd p, $0=\sum_k a_k w_{i_k}(\sum_l(-1)^l w_{(q_k-2l)(p-1)}\rho^l u_{j_k}+\sum_l(-1)^l w_{(q_k-2l)(p-1)-1}\beta\rho^l u_{j_k})$ where $q_k=\deg u_{j_k}$. Consider an element u_{j_k} of the lowest degree, then the right side of the above equality has a leading term $a_k w_{i_k+(p-1)q_k}\times u_{j_k}$. Thus $a_k=0$ for this k, and so on. The case $p=2$ is similar. This shows the above assertion, and Theorem 2 is proved.

3. Computations. Hereafter p denotes an odd prime unless otherwise stated, and $m=(p-1)/2$,

Lemma (Streenrod). *Let u be a q-dimensional cohomology class. Then for any positive integer k, l, q, the following relations hold:*

$$\sum_i(-1)^{i+k}\binom{(q-2i)m}{mq-l+i}\rho^{k-mq+l-i}\rho^i u$$
$$=\sum_i(-1)^{i+l+mq}\binom{(q-2i)^m}{mq-k+i}\rho^{l-mq+k-i}\rho^i u,$$

$$\sum_i(-1)^{i+k+mq+1}\binom{(q-2i)m-1}{mq-l+i}\rho^{k-mq+l-i}\beta\rho^i u$$
$$=\sum_i(-1)^{i+l+1}\binom{(q-2i)m}{mq-k+i}\beta\rho^{k-mq+l-i}\rho^i u$$
$$+\sum_i(-1)^{i+l}\binom{(q-2i)m-1}{mq-k+i}\rho^{k-mq+l-i}\beta\rho^i u.$$

The proof was given in Chapter VIII of [4].

Theorem 3. $\rho^n Pu=\sum_i\binom{(q-2i)m}{n-pi}w_{2(n-pi)(p-1)}P(\rho^i u)$

$$-\mu(q)\sum_i\binom{(q-2i)m-1}{n-pi-1}w_{2(n-pi)(p-1)-p}P(\beta\rho^i u)+z,$$

where $u\in H^q(X;Z_p)$, $z\in\ker d^*$, *and* $\mu(q)$ *denotes* $(m!)^{-1}(-1)^{mq}$.

If $z\in\ker d^*$ then $w_r\times z\in\ker d^*$ for any $r\geq 0$ since d^* is an $H^*(W/\pi;Z_p)$-map. Therefore an easy computation shows

Corollary. $\rho^n(w_s\times Pu)=\sum_i\binom{[s/2]+(q-2i)m}{n-pi}w_{s+2(n-pi)(p-1)}$

$$\times P(\rho^i u)-\mu(q)\varepsilon(s)\sum_i\binom{[s/2]+(q-2i)m-1}{n-pi-1}w_{s-p+2(n-pi)(p-1)}$$
$$\times P(\beta\rho^i u)+z',$$

where $u\in H^q(X;Z_p)$, $z'\in\ker d^*$, $\varepsilon(s)=1$ *if s is even and* $\varepsilon(s)=0$ *if s is odd, and for a real x* $[x]$ *denotes the Gaussian symbol.*

Proof of Theorem 3. Recall that $\beta w_1=w_2$, $w_{2n}=(w_2)^n$, and $\rho^j w_n=\binom{[n/2]}{j}w_{n+2j(p-1)}$. By the definition of the reduced power and by the Cartan formula

$$\rho^n(\nu(q)d^*Pu)=\sum_{i,j}(-1)^i\binom{(q-2i)m}{j}w_{(q-2i+2j)(p-1)}\times\rho^{n-j}\rho^i u$$
$$+\sum_{i,j}(-1)^i\binom{(q-2i)m-1}{j}w_{(q-2i+2j)(p-1)-1}\times\rho^{n-j}\beta\rho^i u.$$

Let $l=mq+i-j$ and $s=n-mq+l-i$, then by the lemma we have

$$\rho^n(\nu(q)d^*Pu)=\sum_l(-1)^n w_{(pq-2l)(p-1)}\sum_i(-1)^{i+n}\binom{(q-2i)m}{mq+i-l}\rho^{n-mq+l-i}\rho^i u$$
$$+\sum_l(-1)^{n+mp+1}w_{(pq-2l)(p-1)-1}\sum_i(-1)^{i+n+mq+1}\binom{(q-2i)m-1}{mq+i-l}\rho^{n-mq+l-i}\beta\rho^i u$$
$$=\sum_l(-1)^n w_{(pq-2l)(p-1)}\sum_i(-1)^{s-n}\binom{(q-2i)m}{mq-n+i}\rho^s\rho^i u$$
$$+\sum_l(-1)^{n+mq+1}w_{(pq-2l)(p-1)-1}\sum_i(-1)^{i+l+1}\binom{(q-2i)m}{mq-n+i}\beta\rho^s\rho^i u$$
$$+\sum_l(-1)^{n+mq+1}w_{(pq-2l)(p-1)-1}\sum_i(-1)^{i+l}\binom{(q-2i)m-1}{mq-n+i}\rho^s\beta\rho^i u,$$

Since $\binom{a}{b}=\binom{a}{a-b}$,

$$=\sum_i w_{2(n-pi)(p-1)}\sum_s(-1)^s\binom{(q-2i)m}{n-pi}w_{(q+2i(p-1)-2s)(p-1)}\rho^s\rho^i u$$
$$+\sum_i w_{2(n-pi)(p-1)}\sum_s(-1)^s\binom{(q-2i)m}{n-pi}w_{(q+2i(p-1)-2s)(p-1)-1}\beta\rho^s\rho^i u$$
$$-\sum_i w_{2(n-pi)(p-1)-p}\sum_s(-1)^s\binom{(q-2i)m-1}{n-pi-1}w_{(q+2i(p-1)+1-2s)(p-1)}\rho^s\beta\rho^i u.$$

If a and b are odd, $w_a\times w_b=0$, thus

$$\rho^n(\nu(q)d^*Pu)=\sum_i w_{2(n-pi)(p-1)}\binom{(q-2i)m}{n-pi}\nu(q+2i(p-1))d^*P(\rho^i u)$$
$$-\sum_i w_{2(n-pi)(p-1)-p}\binom{(q-2i)m-1}{n-pi-1}\nu(q+2i(p-1)+1)d^*P(\beta\rho^i u)\ .$$

Since $\nu(q)^4\equiv 1 \pmod p$, we have $\nu(q+2i(p-1))/\nu(q)\equiv 1 \pmod p$ and $\nu(p+2i(p-1)+1)/\nu(q)\equiv(m!)^{-1}(-1)^{mq}(=\mu(q)) \pmod p$. This completes the proof.

Proof of the main theorem. Let $x_1, x_2, \cdots$ be a canonical basis of $H_*(X;Z_p)$, i.e., homogeneous and if $\Delta x_j\neq 0$ then Δx_j is also a basic element. Denote by $u_1, u_2, \cdots$ the dual cohomology basis. We represent ρ^n_* in matrix forms with respect to this basis, i.e., $\rho^n_* x_i=\sum_j a_{i,j}(n)x_j$, $a_{i,j}(n)\in Z_p$, and by the duality $\rho^n u_j=\sum_k a_{k,j}(n)u_k$. Since d^*P is a homomorphism, we have $P(\rho^n u_j)-\sum_k a_{k,j}(n)Pu_k\in\ker d^*$ and $P(\beta\rho^n u_j)-\sum_k a_{k,j}(n)P(\beta u_k)\in\ker d^*$. Therefore by Theorem 3,

$$\rho^n(w_s\times Pu_j)=\sum_i\binom{[s/2]+(q-2i)m}{n-pi}w_{s+2(n-pi)(p-1)}\sum_k a_{k,j}(i)Pu_k$$
$$-\mu(q)\varepsilon(s)\sum_i\binom{[s/2]+(q-2i)m-1}{n-pi-1}w_{s-p+2(n-pi)(p-1)}\sum_k a_{k,j}(i)P(\beta u_k)+z'.$$

Consider the coefficient of $w_t\times Pu_k$ in $\rho^n(w_s\times Pu_j)$.

Case 1. $u_k\notin\operatorname{Im}\beta$, then the coefficient is

$$\binom{[s/2]+(q_j-2i)m}{n-pi}a_{k,j}(i),\quad\text{where}\quad q_j=\deg u_j.$$

By the duality (Theorem 2), writing $c=t-2n(p-1)$,

$$\rho^n_*(e_{c+2n(p-1)}\otimes_\pi x^p_k)=\sum_{i,j}\binom{[s/2]+(q_j-2i)m}{n-pi}a_{k,j}(i)(e_s\otimes_\pi x^p_j).$$

By the equality of the degrees, $q_k-q_j=2i(p-1)$ and $s=c+2pi(p-1)$, $[s/2]+(q_j-2i)m=[s/2]+(q_k-2ip)m=[c/2]+mq_k$. Therefore we have

$$\rho^n_*(e_{c+2n(p-)}\otimes_\pi x^p_k)=\sum_{i,j}\binom{[c/2]+q_k m}{n-pi}a_{k,j}(i)(e_{c+2i(p-1)}\otimes_\pi x^p_j).$$

Acting $(\Theta_\pi)_*$ on the both sides and using that $Q^{(p)}_i$ is a homomorphism, we have

$$(1)\qquad \rho^n_* Q^{(p)}_{c+2n(p-1)}(x_k)=\sum_i\binom{[c/2]+q_k m}{n-pi}Q^{(p)}_{c+2ip(p-1)}\rho^i_* x_k.$$

Case 2. $u_k=\beta u_l$, then the coefficient is

$$\binom{[s/2]+(q_j-2i)m}{n-pi}a_{k,j}(i)-\mu(q_j)\varepsilon(s')\binom{[s'/2]+(q_j-2i)m-1}{n-pi-1}a_{l,j}(i).$$

Similarly to Case 1,

$$\rho_*^n(e_{c+2n(p-1)}\otimes_\pi x_k^p)=\sum_{i\ j}\binom{[s/2]+(q_j-2i)m}{n-pi}a_{k,j}(i)(e_s\otimes_\pi x_j^p)$$
$$-\sum_{i\ j}(q_j)\varepsilon(s')\binom{[s'/2]+(q_j-2i)m-1}{n-pi-1}a_{l,j}(i)(e_{s'}\otimes_\pi x_j^p).$$

The first summation can be computed as above. Consider the second one, where $pq_k+c=pq_j+s'$, $q_k-q_j=2i(p-1)+1$. So, $s'=c+p+2pi(p-1)$, $[s'/2]+(q_j-2i)m-1=[(c+1)/2]+(q_km-1)$ and $\nu(q_j)\equiv\nu(q_{k+1})\ (\mathrm{mod}\ p)$. $\varepsilon(s')=\varepsilon(c+1)$ since s' and c have an opposite parity. Therefore

$$\rho_*^n(e_{c+2n(p-1)}\otimes_\pi x_k^p)=\sum_{i,j}\binom{[c/2]+q_km}{n-pi}a_{k,j}(i)(e_{c+2ip(p-1)}\otimes_\pi x_j^p)$$
$$-\mu(q_k+1)\varepsilon(c+1)\sum_{i,j}\binom{[(c+1)/2]+q_km-1}{n-pi-1}a_{l,j}(i)(e_{c+p+2ip(p-1)}\otimes_\pi x_j^p).$$

Remark that $x_l=\Delta x_k$. Then we have

$$(2)\qquad \rho_*^nQ_{c+2n(p-1)}^{(p)}x_k=\sum_i\binom{[c/2]+q_km}{n-pi}Q_{c+2ip(p-1)}^{(p)}\rho_*^i(x_k)$$
$$-\mu(q_k+1)\varepsilon(c+1)\sum_i\binom{[(c+1)/2]+mq_k-1}{n-pi-1}Q_{c+p+2ip(p-1)}^{(p)}\rho_*^i\Delta(x_k).$$

If $u_k\notin \mathrm{Im}\,\beta$, then $\Delta x_k=0$. Therefore the formula (1) and (2) coincide, and we have in general

$$\rho_*^nQ_{c+2n(p-1)}^{(p)}x=\sum_i\binom{[c/2]+qm}{n-pi}Q_{c+2ip(p-1)}^{(p)}\rho_*^ix$$
$$-\mu(q+1)\varepsilon(c+1)\sum_i\binom{[(c+1)/2]+qm-1}{n-pi-1}Q_{c+p+2ip(p-1)}^{(p)}\rho_*^i\Delta x.$$

where $x\in H_q(X;Z_p)$. Then the main theorem is an easy restatement of the above formula. For the case $p=2$, the proof is similar and ommited.

References

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