60. On Some Mixed Problems for Fourth Order Hyperbolic Equations

By Sadao MIYATAKE
Department of Mathematics Kyoto, University

(Comm. by Kinjirô KUNUGI, M. J. A., April 12, 1968)

§1. Introduction. We consider some mixed problems for fourth order hyperbolic equations. Let S be a smooth and compact hypersurface in R^n and Ω be the interior or exterior of S. Let

$$\text{(E)}\quad Lu=\left(\frac{\partial^4}{\partial t^4}+(a_1+a_2+a_3)\frac{\partial^2}{\partial t^2}+a_3a_1\right)u+B\left(x,\, t,\, \frac{\partial}{\partial t},\, D\right)u=f.$$

Here $a_k(k=1, 2, 3)$ are the following operators:

$$a_k=-\sum_{ij}^n \frac{\partial}{\partial x_i}\left(a_{k,ij}(x)\frac{\partial}{\partial x_j}\right)+b_k(x, D),$$

$$(1.1)\qquad a_{k,ij}(x)=a_{k,ji}(x) \text{ are real,}$$

$$\sum_{ij}^n a_{k,ij}(x)\xi_i\xi_j\geq\delta|\xi|^2,\ (\delta>0)$$

$$\text{for every } (x, \xi)\in\Omega\times R^n\ (k=1, 2, 3),$$

B denotes an arbitrary third order differential operator and b_k are first order operators. Let us assume that all coefficients are sufficiently differentiable and bounded in $\bar{\Omega}$ or in $\bar{\Omega}\times(0, \infty)$.

Recently S. Mizohata [1] treated mixed problems for the equations of the form

$$L=\prod_{i=1}^m\left(\frac{\partial^2}{\partial t^2}+c_i(x)a(x, D)\right)+B_{2m-1},\quad c_i(x)>c_{i+1}(x),\quad c_i(x)>0$$

$$(i=1, \cdots, m).$$

Let us consider the case where $m=2$. The above equation has the form

$$\frac{\partial^4}{\partial t^4}+(c_1(x)+c_2(x))a\frac{\partial^2}{\partial t^2}+c_1c_2a^2+(\text{operator of third order}).$$

Now it is not difficult to see that this operator can be considered as a special class of (E), by putting $a_1=\alpha c_1 a$, $a_2=(1-\alpha)c_1 a+\left(1-\frac{1}{\alpha}\right)c_2 a$, α being a constant less than 1 chosen closely to 1. We consider the case where the operators a_k have some relations only at the boundary. Let us denote the Sobolev space $H^p(\Omega)$ simply by H^p, and its norm by $\|\cdot\|_p$ and denote the closure of $\mathcal{D}(\Omega)$ in H^1 by $\mathcal{D}^1_{L^2}$. Define

$$D(a_k)=\{u\in H^3\cap\mathcal{D}^1_{L^2};\, a_k u\in\mathcal{D}^1_{L^2}\}.$$

Namely, $u\in H^3$ belongs to $D(a_k)$ means that not only u itself but also

$a_k u$ vanish at the boundary. We *assume* that

(H) $D(a_1)=D(a_2)=D(a_3) \quad (=D(a))$.

Our boundary conditions are followings:

(Case I) $\quad u|_s=0, \quad a_1 u|_s=0$

(Case II) $\quad \left(\frac{\partial}{\partial n_1}+\sigma(s)\right)u|_s=0, \quad \left(\frac{\partial}{\partial n_1}+\sigma(s)\right)a_1 u|_s=0,$

where $$\frac{\partial}{\partial n_1}=\sum_{ij} a_{1,ij}(x)\cos(\nu, x_j)\frac{\partial}{\partial x_i}, \qquad (\nu\text{ ; outer normal}),$$

and $\sigma(s)$ is a smooth complex-valued function defined on S.

Consider the case where $B\equiv 0$. Put

(1.2) $$u_0=u, \quad u_1=\frac{\partial}{\partial t}u, \quad u_2=\frac{\partial^2 u}{\partial t^2}+a_1 u, \quad u_3=\frac{\partial^3}{\partial t^3}u+(a_1+a_2)\frac{\partial}{\partial t}u.$$

Then the equation (E) with $B\equiv 0$ is reduced to

(1.3) $$\frac{d}{dt}U(t)=AU(t)+F(t),$$

where $U(t)={}^t(u_0(t), u_1(t), u_2(t), u_3(t))$, $F(t)={}^t(0, 0, 0, f(t))$, and

(1.4) $$A=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -a_1 & 0 & 1 & 0 \\ 0 & -a_2 & 0 & 1 \\ 0 & 0 & -a_3 & 0 \end{pmatrix}.$$

Conversely if $U(t)$ satisfies (1.2), then $u_0(x, t)$ satisfies (E) with $B\equiv 0$. Let us denote

$$N=\left\{u\in H^2;\ \left(\frac{\partial}{\partial n_1}+\sigma\right)u|_s=0\right\}.$$

We introduce two Hilbert spaces according to Case I and Case II.

(1.5) $$\mathcal{H}_1=D(a)\times H^2\cap\mathcal{D}^1_{L^2}\times\mathcal{D}^1_{L^2}\times L^2$$
$$\mathcal{H}_2=H^3\cap N\times N\times H^1\times L^2.$$

These spaces are closed subspaces of $H^3\times H^2\times H^1\times L^2$ equipped with the canonical norm

(1.6) $$\|U\|^2=\|u_0\|_3^2+\|u_1\|_2^2+\|u_2\|_1^2+\|u_3\|_0^2.$$

According to Cases I and II, we take the definition domains of A as follows

(1.7) $$D(A)_1=H^4\cap D(a)\times D(a)\times H^2\cap\mathcal{D}^1_{L^2}\times\mathcal{D}^1_{L^2}$$
$$D(A)_2=N(a_1)\times H^3\cap N\times N\times H^1, \text{ where}$$

(1.8) $$N(a_1)=\{u\,;\,u\in H^4\cap N,\ a_1 u\in N\}.$$

For convenience we note for $U\in D(A)_i \quad (i=1, 2)$

(1.9) $$\|U\|^2_{D(A)i}=\|u_0\|_4^2+\|u_1\|_3^2+\|u_2\|_2^2+\|u\|_1^2.$$

$D(A)_1$ and $D(A)_2$ are dense in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. In fact, in view of the regularity theorem on elliptic boundary problems, we can show easily that $D(a)$ is dense in $\mathcal{D}^1_{L^2}\cap H^2$, and that $N(a_1)$ is dense in $N\cap H^3$.

Now we state our result.

Theorem. *For any $f(t)$ in $\mathcal{E}_t^1(L^2)$*[1] *and any initial data* $(u(x,0), \frac{\partial}{\partial t}u(x,0), \frac{\partial^2}{\partial t^2}u(x,0), \frac{\partial^3}{\partial t^3}u(x,0))$ *in $D(A)_i$, there exists a unique solution of the equation (E), satisfying the boundary condition (I) or (II). The solution $U(t)$ is in $\mathcal{E}_t^1(\mathcal{H}_i)\cap\mathcal{E}_t^0(D(A)_i)$. Moreover when we assume the compatibility condition on the initial data and the regularity of $f(t)$, then the solution has the same regularity as the initial data.*

§2. Some lemmas. Let $\Phi(x)$ be the distance from x to the surface measured along a straight line issuing from S with the conormal direction. For $a=-\sum\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)+$(first order operator) put

$$\alpha(x)=\sum a_{ij}(x)\frac{\partial\Phi(x)}{\partial x_i}\frac{\partial\Phi(x)}{\partial x_j}. \tag{2.1}$$

Lemma 1. *(Decomposition of second order elliptic operators). Assume that a satisfies (1.1), then a is written in $\bar{\Omega}$ in the following form:*

$$a=n^*(x,D)n(x,D)-\sum_{j:\text{ finite}}t_j(x,D)s_j(x,D)+(\textit{first order term}). \tag{2.2}$$

Here t_j and s_j are first order operators and tangential on S. The operator n has the following form:

$$n(x,D)=\frac{\zeta(x)}{\sqrt{\alpha(x)}}\sum_{ij}^{n}a_{ij}(x)\left(-\frac{\partial\Phi}{\partial x_j}(x)\right)\frac{\partial}{\partial x_i} \tag{2.3}$$

where $\zeta(x)$ is a C^∞-function taking the value 1 in a small neighborhood of S, and vanishing outside of some neighborhood of S.

Remark. We say that a first differential operator is tangential at the boundary S, if

$$t(x,D)=\sum c_j(x)\frac{\partial}{\partial x_j}+d(x)$$

satisfies $\sum c_j(x)\cos(\nu,x_j)=0$, for all $x\in S$. Then we have the following relation:

$$(t(x,D)u(x),v(x))=(u(x),t^*(x,D)v(x)) \text{ for all } u,v\in H^1.$$

Lemma 2. 1) $\frac{\partial}{\partial n_i}=\beta_i(x)\frac{\partial}{\partial n_1}$, $\left(\beta_i(x)=\frac{\alpha_i}{\alpha_1}\right)$, $(i=2,3)$, $x\in S$.

2) *If $u\in H^3$ vanishes at the boundary, then $(a_3-\beta_3a_1)u$ vanishes also at the boundary.*

Sketch of the proof. After a local transformation, let

$$a_i=b_i\left(x,y,D_x,\frac{\partial}{\partial y}\right)\frac{\partial}{\partial y}+c_i(x,y,D_x)\ (i=1,2), \tag{2.4}$$

---

1) $f(t)\in\mathcal{E}_t^p(H)$ $(p=0,1,2,\dots)$ means that $f(t)$ is p times continuously differentiable in t with values in H.

where $b_i\left(x, y, D_x, \frac{\partial}{\partial y}\right)$ are first order operators and $c_i(x, y, D_x)$ do not contain $\frac{\partial}{\partial y}$. Then, a_i $(i=1, 2)$ satisfy (H) if and only if the following relation holds

$$b_2\left(x, 0, D_x, \frac{\partial}{\partial y}\right)=\beta_2(x)b_1\left(x, 0, D_x, \frac{\partial}{\partial y}\right). \tag{2.5}$$

Lemma 3. *Assume that a_1 and a_2 satisfy* (1.1), *then there exists a positive constant δ such that for sufficiently large constant r,*

$$Re(a_1u, a_2u)+r\|u\|_0^2\geq\delta\|u\|_2^2 \text{ for all } u\in H^2\cap\mathcal{D}_{L^2}^1$$

or for $u\in N$.

Lemma 4. *Assume that a_1, a_2, and a_3 satisfy* (1.1) *and* (H), *then we have*

$$Re\sum_{ij}^{n}\left(a_{3,ij}(x)\frac{\partial}{\partial x_i}a_1u, \frac{\partial}{\partial x_j}a_2u\right)+r\|u\|_0^2\geq\delta\|u\|_3^2$$

for all $u\in D(a)$ or $u\in H^3\cap N$.

Lemma 5. *Under the same assumption as in Lemma* 4, *there exists a positive constant C such that*

$$|(a_1u, a_2v)-(a_2u, a_1v)|\leq C\|u\|_2\|v\|_1$$

for all $u\in H^3\cap N$, $v\in N$, or for $u\in D(a)$ and $v\in H^2\cap\mathcal{D}_{L^2}^1$.

Lemma 6. *Under the same assumption as in Lemma* 5, *we have*

$$|(a_1u_1, a_2a_3u_0)-(a_2u_1, a_1a_3u_0)|\leq C\|u_1\|_2\|u_0\|_3$$

for all $u_0\in N(a_1)$ and $u_1\in H^3\cap N$.

§3. Evolution equation and existence of solutions. We introduce the following hermitian form in $\mathcal{H}_1$ defined by

$$\begin{aligned}(U, V)_{\mathcal{H}_1}=&\sum_{ij}^{n}\left\{\left(a_{2,ij}(x)\frac{\partial}{\partial x_j}a_1u_0, \frac{\partial}{\partial x_i}a_3v_0\right)\right.\\&\left.+\left(a_{2,ij}(x)\frac{\partial}{\partial x_j}a_3u, \frac{\partial}{\partial x_i}a_1v_0\right)\right\}+r(u_0, v_0)\\&+\{(a_2u_1, a_3v_1)+(a_3u_1, a_2v_1)+r(u_1, v_1)\}\\&+\left\{2\sum_{ij}^{n}\left(a_{3,ij}(x)\frac{\partial}{\partial x_j}u_2, \frac{\partial}{\partial x_i}v_2\right)+r(u_2, v_2)\right\}+2(u_3, v_3).\end{aligned} \tag{3.1}$$

In Case II we use the hermitian form of the following type:

$$\begin{aligned}(U, V)_{\mathcal{H}_2}=&[u_0, v_0]+\{(a_2u_1, a_3v_1)+(a_3u_1, a_2v_1)+r(u_1, v_1)\}\\&+\left\{2\sum_{ij}^{n}\left(a_{3,ij}(x)\left(\frac{\partial}{\partial x_j}+\sigma_j\right)u_2, \left(\frac{\partial}{\partial x_i}+\sigma_i\right)v_2\right)+r(u_2, v_2)\right\}\\&+2(u_3, v_3).\end{aligned} \tag{3.2}$$

It would be natural to take the following hermitian form for $[u_0, v_0]$:

$$\begin{aligned}&((n_2+\rho)a_1u_0, (n_2+\rho)a_3v_0)+((n_2+\rho)a_3u_0, (n_2+\rho)a_1v_0)\\&+\sum_j(t_{2j}a_1u_0, s_{2j}a_3v_0)+\sum(s_{2j}a_3u_0, t_{2j}a_1v_0)+r(u_0, v_0),\end{aligned}$$

where s_{2j} and t_{2j} are first order tangential operators derived from the decomposition of Lemma 1 with respect to the operator a_2.

However for this form the calculus by integration by parts concerning $(AU, U)_{\mathcal{H}_2}+(U, AU)_{\mathcal{H}_2}$ does not work well. Taking account of the fact that $(a_3-\beta_3 a_1)(n_2+\rho)u_0$ and $(n_2+\rho)a_1 u_0$ vanish at the boundary for $u_0 \in N(a_1)$ (in view of Lemma 2),
we introduce the following hermitian form:

$$[u_0, v_0]=((n_2+\rho)a_1 u_0, \gamma_3(x, D)v_0)+(\gamma_3(x, D)u_0, (n_2+\rho)a_1 v_0) + \sum_j \{(t_{2j}a_1 u_0, s_{2j}a_3 v_0)+(s_{2j}a_3 u_0, t_{2j}a_1 v_0)\}+r(u_0, v_0), \tag{3.3}$$

where

$$\gamma_3(x, D)=(a_3-\beta_3 a_1)(n_2+\rho)+\beta_3(n_2+\rho)a_1. \tag{3.4}$$

Here $\sigma_i(x)$ $(i=1, 2, \cdots, n)$ and $\rho(x)$ appearing in (3.2), (3.3) are arbitrary sufficiently smooth functions satisfying on S the following conditions:

$$\begin{aligned} &\sum_{ij} a_{1,ij}\sigma_j(x)\cos(\nu, x_i)=\sigma(s) \text{ on } S \\ &(\sum_{ij} a_{2,ij}\cos(\nu, x_i)\cos(\nu, x_j))^{\frac{1}{2}}(\sum a_{1,ij}\cos(\nu, x_i)\cos(\nu, x_j))^{-1}\sigma(s) \\ &\quad =\rho(s) \text{ on } S. \end{aligned} \tag{3.5}$$

By virtue of Lemma 3 and Lemma 4, there exists a positive constant C such that

$$\frac{1}{C}\|U\|\leq(U, U)_{\mathcal{H}_i}\leq C\|U\| \qquad (i=1,2) \text{ for } U\in\mathcal{H}_i. \tag{3.6}$$

Considering Lemmas 5 and 6 we obtain the following estimates for another constant C

$$|(AU, U)_{\mathcal{H}_i}+(U, AU)_{\mathcal{H}_i}|\leq C\|U\| \text{ for all } U\in D(A)_i \qquad (i=1, 2). \tag{3.7}$$

Proposition 1. For any $U\in D(A)_i$, there exists a positive number β such that

$$\|(\lambda I-A)U\|_{\mathcal{H}_i}\geq(|\lambda|-\beta)\|U\|_{\mathcal{H}_i} \qquad \text{for } |\lambda|>\beta,\ \lambda \text{ real}. \tag{3.8}$$

Let us show that there exists $U\in D(A)_i$ such that $(\lambda I-A)U=F$ holds for any F in $\mathcal{H}_i$. For this purpose it suffices to prove that there exists $u\in H^4\cap D(a)$ or $u\in N(a_1)$ such that

$$(\lambda^4+(a_1+a_2+a_3)\lambda^2+a_3 a_1)u=g \tag{3.9}$$

holds for any g in L^2 and $|\lambda|>\beta$. This is reduced to the theory of the elliptic boundary value problems containing a real parameter (c.f. S. Mizohata [1]).

Thus we are in a position to apply Hille-Yosida's theorem.

Proposition 2. When we assume $F(t)\in\mathcal{E}_t^0(D(A)_i)$ and the initial data $U(0)\in D(A)_i$, then we have a unique solution in $\mathcal{E}_t^1(\mathcal{H}_i)\cap\mathcal{E}_t^0(D(A)_i)$ of the equation (1.3) represented by

$$U(t)=T_t U(0)+\int_0^t T_{t-s}F(s)ds, \tag{3.10}$$

where T_t is the semi-group with the infinitesimal generator A.

Moreover we have the following energy inequality:

Proposition 3. Assume that $f(t)$ is in $\mathcal{E}_t^1(L^2)$, then we have

$$\|U(t)\|_{D(A)_i}+\left\|\frac{\partial^4}{\partial t^4}u(t)\right\|_0\leq C(T)\left\{\|U(0)\|_{D(A)_i}+\|f(0)\|_0 + \int_0^t \|f'(t)\|_0 dt\right\}, \qquad 0\leq t<T,$$

for the solutions $U(t)\in\mathcal{E}_t^0(D(A)_i)\cap\mathcal{E}_t^1(\mathcal{H}_i)$ of the equation (1.3).

By Propositions 2 and 3, we can use the method of successive approximation to the equation (E). Thus we arrive at the Theorem stated in §1.

References

[1] S. Mizohata: Quelques problèmes au bord, du type mixte, pour des équations hyperboliques. Collège de France, 23–60 (1966–67).

[2] M. Schechter: General boundary value problems for elliptic equations. Comm. Pure Appl. Math., **19**, 457–486 (1959).

[3] K. Yosida: An operator theoretical integration of the wave equations. J. Math. Soc. Japan, **8**, 77–92 (1956).