59. A Remark on Baire's Theorem in Ranked Spaces

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The Baire's theorem in ranked spaces was studied by Prof. K. Kunugi in 1954. He showed then that in a topological space which is at the same time a complete ranked space the Baire's theorem holds ([1] p. 556). To be exact;

Theorem. *If a topological space E is a complete ranked space with indicator θ, then, for any well-ordered sequence*

$$G_0, G_1, \cdots, G_\alpha, \cdots ;\ 0 \leq \alpha < \theta$$

of open[1] *and everywhere dense*[1] *subsets in E, $\bigcap_\alpha G_\alpha$ is also everywhere dense*[1] *in E.*

This theorem is a generalization of Baire's theorem which states that every complete metric space, or every locally compact regular space is a Baire space ([1] p. 912).

In this note we shall show the existence of a complete ranked space in which Baire's theorem does not hold.

One of such spaces is the space $\mathcal{D}$, defined by L. Schwartz, consisting of all infinitely differentiable function φ with compact carrier ([3], [4] p. 587). We shall treat the case in which φ is of one variable, and use the same notation as that in the note [4].

First, we shall show the completeness of the ranked space $\mathcal{D}$. Let $\{\varphi_\nu + v(n_\nu, K_\nu; 0)\}_{\nu=0,1,2,\ldots}$ be a fundamental sequence of neighbourhoods in $\mathcal{D}$ ([6] p. 251). Then we have:

(i) $K_0 \geq K_1 \geq K_2 \geq \cdots \geq K_\nu \geq \cdots$;

(ii) car. $\varphi_{\nu+1} \subseteq$ car. $\varphi_\nu \cup [-K_\nu, K_\nu]$;

(iii) for each fixed non-negative integer n, $\{\varphi^{(n)}(x)\}_{\nu=0,1,2,\ldots}$[2] converges uniformely to a continuous function $\varphi_n(x)$ on $(-\infty, \infty)$.

Therefore, there is a function φ in $\mathcal{D}$ of which $\varphi^{(n)}(x) = \varphi_n(x)$ $(n = 0, 1, 2, \cdots)$. It is easily seen that

$$\varphi \in \bigcap_\nu (\varphi_\nu + v(n_\nu, K_\nu; 0)).$$

Hence, $\mathcal{D}$ is complete.

Next, for any non-negative integer K, let $\mathcal{D}_K$ be the totality of φ in $\mathcal{D}$ of which the carrier is contained in interval $[-K, K]$, and C_K be the compliment of $\mathcal{D}_K$ in $\mathcal{D}$. Then $\mathcal{D}_K$ is r-closed ([5] p. 69) and

1) In the topological sense.

2) $\varphi^{(n)}$ denotes the n-th derivative of φ, and $\varphi^{(0)}$ means φ.

$\mathcal{C}_K$ is r-open.[3] And $\mathrm{cl}(\mathcal{C}_K)=\mathcal{D}$, i.e. $\mathcal{D}_K$ is nowhere dense.

Therefore $\mathcal{D}=\bigcup_K \mathcal{D}_K$, $\mathcal{D}$ is of the 1st category.

References

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