57. *On the Ranked Group*

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The purpose of this note is to give a definition of the ranked group, i.e. to combine the notions of the group and the ranked space [1], by the same method in the definition of the topological group. Throughout this note, we shall treat only ranked spaces with indicator ω_0. We shall denote the points of a ranked space by $x, y, \cdots$, the family of neighbourhoods of x with rank n by $\mathfrak{V}_n(x)$, and fundamental sequences of neighbourhoods with respect to x[1] by $\{u_n(x)\}$, $\{v_n(x)\}, \cdots$.

§ 1. The definition of ranked groups. A set G is called a ranked group, if it is a group which is also a ranked space, where the group operations $(x, y)\to xy$, $x\to x^{-1}$, are continuous in the following sense;

(I) for any $\{u_n(x)\}$, $\{v_n(x)\}$, there exists a $\{w_n(xy)\}$ such that $u_n(x)v_n(y)\subseteq w_n(xy)$

(II) for any $\{u_n(x)\}$, there exists a $\{v_n(x^{-1})\}$ such that $(u_n(x))^{-1}\subseteq v_n(x^{-1})$.

(I) implies that, if $\{\lim x_n\}\ni x$ and $\{\lim y_n\}\ni y$, then $\{\lim x_n y_n\}\ni xy$, (II) implies that, if $\{\lim x_n\}\ni x$, then $\{\lim x_n^{-1}\}\ni x^{-1}$.

§ 2. The neighbourhoods of identity of a ranked group. Let G be a ranked group, and e be its identity. $\mathfrak{V}_n$ will denote the family of neighbourhoods of e with rank n, and $\{U_n\}, \{V_n\}, \cdots$ fundamental sequences of neighbourhoods with respect to e.

The system $\{\mathfrak{V}_n\}$ possesses the following properties:

(A) for every V in $\mathfrak{V}$, $e\in V$ (where $\mathfrak{V}=\bigcup_{n=0}^{\infty}\mathfrak{V}_n$)

(B) for any U, V in $\mathfrak{V}$, there is a W in $\mathfrak{V}$ such that $U\cap V\subseteq W$

(a) for any V in $\mathfrak{V}$ and for any integer n, there is an m, $m\geq n$, and a U in $\mathfrak{V}_m$ such that $U\subseteq V$

(b) $G\in\mathfrak{V}_0$.

These are obvious as the properties of neighbourhoods in a ranked space.

The axioms (I), (II) yields also,

(RG_1) for any $\{U_n\}$, $\{V_n\}$, there is a $\{W_n\}$ such that $U_nV_n\subseteq W_n$,

(RG_2) for any $\{U_n\}$, there is a $\{V_n\}$ such that $U_n^{-1}\subseteq V_n$,

1) A sequence of neighbourhoods of x, $\{v_n(x)\}$, is called a fundamental sequence, if $v_n(x)\supseteq v_{n+1}(x)$, and $\alpha_n\uparrow\infty$, where α_n is the rank of $v_n(x)$.

(RG_3) for any $\{U_n\}$ and for any $x \in G$, there is a $\{V_n\}$ such that $xU_nx^{-1} \subseteq V_n$,

(RG_4l) (resp. (RG_4r)) Let x be any point of G. For any $\{U_n\}$ there is a $\{v_n(x)\}$ such that $xU_n \subseteq v_n(x)$ (resp. $U_nx \subseteq v_n(x)$), and, conversely, for any $\{u_n(x)\}$ there is a $\{V_n\}$ such that $u_n(x) \subseteq xV_n$ (resp. $u_n(x) \subseteq V_nx$).[2)]

Proof. (RG_1), (RG_2) are immediate consequences of (I), (II), respectively, putting $x=y=e$. We shall prove (RG_4l). Let $\{u_n(x)\}$ be some fundamental sequence of neighbourhoods with respect to x. Because of (I), there exists a $\{v_n(x)\}$ such that $u_n(x)U_n \subseteq v_n(x)$. Since $x \in u_n(x)$, $xU_n \subseteq v_n(x)$. Conversely, taking some fundamental sequence of neighbourhoods with respect to x^{-1}, say $\{v_n(x^{-1})\}$, and applying (I), there exists a $\{V_n\}$ such that $v_n(x^{-1})u_n(x) \subseteq V_n$. Since $x^{-1} \in v_n(x^{-1})$, $x^{-1}u_n(x) \subseteq V_n$, i.e. $u_n(x) \subseteq xV_n$.

Similarly we can prove (RG_4r).

Now, we shall prove (RG_3). For any $\{U_n\}$ and for any $x \in G$, because of (RG_4l), we get a $\{v_n(x)\}$ such that $xU_n \subseteq v_n(x)$. Then, from (RG_4r), there exists a $\{V_n\}$ such that $v_n(x) \subseteq V_nx$. Hence $xU_nx^{-1} \subseteq V_n$.

The four conditions above are not only necessary, but sufficient for a group G which is also a ranked space to be a ranked group. In other words, (I), (II) follows from (RG_1), (RG_2), (RG_3), (RG_4l) (or (RG_4r)).[3)] Clearly, (RG_3) can be omitted if G is commutative.

Proof. (I). Take any $\{u_n(x)\}$, $\{v_n(y)\}$. From (RG_4l) and (RG_4r), there are $\{U_n\}$, $\{V_n\}$ such that $u_n(x) \subseteq xU_n$, $v_n(y) \subseteq V_ny$. Applying (RG_1), we get a $\{W_n\}$ such that $U_nV_n \subseteq W_n$, and furthermore, by (RG_3), a $\{W_n'\}$ such that $xW_nx^{-1} \subseteq W_n'$. From (RG_4r) again, there is a $\{w_n(xy)\}$ such that $W_n'xy \subseteq w_n(xy)$. Then, $u_n(x)v_n(y) \subseteq xU_nV_ny \subseteq xW_ny \subseteq W_n'xy \subseteq w_n(xy)$.

(II) By (RG_4l), for any $\{u_n(x)\}$, there is a $\{U_n\}$ such that $u_n(x) \subseteq xU_n$. Next, by (RG_2) there is a $\{V_n\}$ such that $U_n^{-1} \subseteq V_n$, and by (RG_4r), a $\{v_n(x^{-1})\}$ such that $V_nx^{-1} \subseteq v_n(x^{-1})$. Then, $(u_n(x))^{-1} \subseteq (xU_n)^{-1} \subseteq V_nx^{-1} \subseteq v_n(x^{-1})$.

Now, let G be a group, where defined families of subsets, $\mathfrak{V}_n (n=0,1,2,\cdots)$, which satisfy axioms (A), (B), (a), (b), (RG_1), (RG_2), (RG_3). When we take the totality of xV for $V \in \mathfrak{V}_n$ as $\mathfrak{V}_n(x)$, (RG_4l) is evidently fulfilled, and G becomes a ranked group.[4)] Taking $\{Vx\,;\,V \in \mathfrak{V}_n\}$ as

2) (RG_4l) (resp. (RG_4r)) means that, in a ranked group, we have $\{\lim x_n\} \ni x$ if and only if $\{\lim x^{-1}x_n\} \ni e$ (resp. $\{\lim x_nx^{-1}\} \ni e$).

3) Under the condition (RG_3), conditions (RG_4l) and (RG_4r) are equivalent.

4) If G is a ranked group and $\mathfrak{V}_n(x)$ is the system of neighbourhoods of x with rank n, taking the new system $\mathfrak{V}_n'(x)=\{xV;\ V \in \mathfrak{V}_n\}$, a new ranked group may be obtained, where convergence of sequences coincides with initial one. See examples [2], [3].

$\mathfrak{B}_n(x)$, we may obtain another ranked group. In any case convergence of sequences coincides.

§ 3. Sufficient conditions for (RG_1), (RG_2), (RG_3). As sufficient conditions for (RG_1), (RG_2), (RG_3), respectively, we have

(1) there exists a non-negative function $\phi(\lambda, \mu)$, defined for $\lambda \geq 0$, $\mu \geq 0$, such that $\lim_{\lambda, \mu \to \infty} \phi(\lambda, \mu) = \infty$, and the following holds; if $U \in \mathfrak{B}_l$, $V \in \mathfrak{B}_m$, $W \in \mathfrak{B}_n$, and $UV \subseteq W$, then there is an $n^* \geq \phi(l, m)$, and a W^* in $\mathfrak{B}_{n^*}$ such that $UV \subseteq W^* \subseteq W$.

(2) there exists a non-negative function $\psi(\lambda)$ defined for $\lambda \geq 0$ such that $\lim_{\lambda \to \infty} \psi(\lambda) = \infty$, and the following holds; if $U \in \mathfrak{B}_l$, $V \in \mathfrak{B}_m$, and $U^{-1} \subseteq V$, then there is an $m^* \geq \psi(l)$ and a V^* in $\mathfrak{B}_{m^*}$ such that $U^{-1} \subseteq V^* \subseteq V$.

(3) there exists a non-negative function $\chi(\lambda; x)$ defined for $\lambda \geq 0$, $x \in G$, such that $\lim_{\lambda \to \infty} \chi(\lambda; x) = \infty$ for any fixed x, and the following holds; if $U \in \mathfrak{B}_m$, $V \in \mathfrak{B}_n$, $x \in G$, and $xUx^{-1} \subseteq V$, there is an $n^* \geq \chi(m; x)$ and a V^* in $\mathfrak{B}_{n^*}$ such that $xUx^{-1} \subseteq V^* \subseteq V$.

The proof can be made by the same method in [2].

When $\{\mathfrak{B}_n\}$ satisfies the condition $(*)$ ([2], p. 586), (1), (2), (3) may be replaced by, respectively,

(1′) there exists a function $\phi(\lambda, \mu)$ such as ϕ in (1), and the following holds; for any $U \in \mathfrak{B}_l$, $V \in \mathfrak{B}_m$, there is an $n \geq \phi(l, m)$ and a W in $\mathfrak{B}_n$ such that $UV \subseteq W$.

(2′) there exists a function $\psi(\lambda)$ such as ψ in (2), and the following holds; for any $U \in \mathfrak{B}_l$ there is an $m \geq \psi(l)$ and a V in $\mathfrak{B}_m$ such that $U^{-1} \subseteq V$.

(3′) there exists a function $\chi(\lambda; x)$ such as χ in (3), and the following holds; for any $U \in \mathfrak{B}_m$ and for any $x \in G$, there is an $n \geq \chi(m; x)$ and a V in $\mathfrak{B}_n$ such that $xUx^{-1} \subseteq V$.

§ 4. Limit structure on ranked groups. H. R. Fischer developed the theory of limit spaces, and generalized the notion of topological group to group with limits [3]. A ranked group is considered to be a group with limits as follows.

We take as τe the families of all filters containing some fundamental sequence of neighbourhoods with respect to e. It is easy to see that the ultrafilter $\dot{e}$ the (family of all subsets containing e) belongs to τe, and that τe is a Λ-ideal, i.e. i) if $\mathfrak{F} \in \tau e$ and $\mathfrak{G} \in \tau e$, then $\mathfrak{F} \wedge \mathfrak{G} \in \tau e$, ii) if $\mathfrak{F} \in \tau e$ and $\mathfrak{F} \leq \mathfrak{G}$, then $\mathfrak{G} \in \tau e$. Moreover, τe satisfies following conditions; (I) $\tau e \cdot \tau e \subseteq \tau e$ (II) $(\tau e)^{-1} \subseteq \tau e$, (III) $x \cdot \tau e \cdot x^{-1} \subseteq \tau e$. In fact, these follows from (RG_1), (RG_2), (RG_3), respectively. Thus, G becomes a group with limits ([3], p. 293, Satz 1).

A sequence $\{x_n\}$ converges to x in the limit space G if and only if

$\{\lim x_n\} \ni x$. In fact,[5] $\{x_n\}$ converges to e in the limit space G if and only if the filter $\mathfrak{A}$ generated by all A_n's, $A_n=\{x_k; k \geq n\}$, belongs to τe. If $\{\lim x_n\} \ni e$, there is a $\{V_n\}$ such that $x_n \in V_n$. Since $V_n \supseteq V_{n+1}$, $V_n \supseteq A_n$ and therefore $V_n \in \mathfrak{A}$. Hence $\mathfrak{A} \in \tau e$. Conversely, let $\mathfrak{A} \in \tau e$. There is a $\{U_n\}$ such that $U_n \in \mathfrak{A}$. Since $\mathfrak{A}$ is generated by A_n's and $A_n \supseteq A_{n+1}$, there is an N_1 such that $U_1 \supseteq A_{N_1}$, and an $N_2 > N_1$ such that $U_2 \supseteq A_{N_2}$, and so on. Putting $V_n = U_i$ for n with $N_i \leq n < N_{i+1}$ (where $N_0=1$, $V_0=G$), we get a fundamental sequence $\{V_n\}$ such that $V_n \supseteq A_n$, therefore $x_n \in V_n$, that is, $\{\lim x_n\} \ni e$.

§ 5. Examples of ranked groups. 1. A linear ranked space [2] is an additive ranked group. In fact, axiom (1) in linear ranked spaces is identical to the condition (1). (RG_2) is satisfied because all V in $\mathfrak{V}$ are circled, and (RG_4l) is trivial, for $\mathfrak{V}_n(x)$ is defined as the family of all $x+V$ for $V \in \mathfrak{V}_n$.

2. Let B be a Banach algebra, G be the group consisting of all regular points of B. Take as $\mathfrak{V}_n(x)$ the family consisting of only one set $v(n; x) = \left\{y \in G; \| y-x \| < \frac{1}{n}\right\}$ $(n=1, 2, \cdots)$, and $\{G\}$ as $\mathfrak{V}_0(x)$. Then G is clearly a ranked space. Since $v(m; e) \subseteq v(n; e)$ for $m \geq n$, the condition (*) is satisfied. Put $\phi(\lambda, \mu) = \left[\frac{\lambda\mu}{\lambda+\mu+1}\right]$. Then, $\lim\limits_{\lambda, \mu \to \infty} \phi(\lambda, \mu) = \infty$, and we can show that $v(l; e)v(m; e) \subseteq v(n; e)$ for $n=\phi(l, m)$, from the inequality $\| xy \| \leq \| x \| \cdot \| y \|$. Consequently (1′) holds. Next, putting $\psi(\lambda) = \max.(\lambda-1, 0)$, we have $\lim\limits_{\lambda \to \infty} \psi(\lambda) = \infty$, and $(v(l; e))^{-1} \subseteq v(m; e)$ for $m=\psi(l)$.[6] Finally, (3′) also holds, putting $\chi(\lambda; x) = \left[\frac{\lambda}{\| x \| \cdot \| x^{-1} \|}\right]$. Moreover ($RG_4l$) is satisfied. In fact, since $xv(m; e) \subseteq v(n; x)$ for $n = \left[\frac{m}{\| x \|}\right]$, and $v(m; x) \subseteq xv(n; e)$ for $n = \left[\frac{m}{\| x^{-1} \|}\right]$, for every $\{u_n(x)\}$, we can choose a $\{V_n\}$ such that $u_n(x) \subseteq xV_n$, and, conversely, for every $\{U_n\}$, a $\{v_n(x)\}$ such that $xU_n \subseteq v_n(x)$. Thus, G is a ranked group.

When we take $\{xv(n; e)\}$ as $\mathfrak{V}_n(x)$, G becomes a new ranked group, in which convergence is still the norm convergence, but the diameter of the neighbourhood of x with rank n may depend upon x.

3. Let G be the group consisting of all matrices of the form $\begin{pmatrix} a_1, & a_2 \\ 0, & 1 \end{pmatrix}$

5) Since, both in ranked groups and in groups with limits, a sequence $\{x_n\}$ converges to x if and only if $\{x^{-1}x_n\}$ converges to e, we can assume $x=e$ without loss of generality.

6) For the proof, remark that, since $x^{-1} = e + \sum\limits_{n=1}^{\infty} (x-e)^n$ for x such that $\| x-e \| < 1$, $\| x-e \| < \frac{1}{n}$ yields $\| x^{-1}-e \| < \frac{1}{n-1}$ $(n \geq 2)$.

$(0<a_1<\infty, -\infty<a_2<\infty)$.[7] We denote $\begin{pmatrix} a_1, a_2 \\ 0, 1 \end{pmatrix}$ by simply (a_1, a_2). Then $e=(1, 0)$, and if $a=(a_1, a_2)$, $b=(b_1, b_2)$, then $ab=(a_1b_1, a_1b_2+a_2)$, $a^{-1} = \left(\frac{1}{a_1}, -\frac{a_2}{a_1}\right)$. This group is not commutative.

Let $v(n\,;a)=\left\{x=(x_1, x_2)\in G\,;(x_1-a_1)^2+(x_2-a_2)^2<\frac{1}{n^2}\right\}$, and take $\{v(n\,;a)\}$ as $\mathfrak{B}_n(a)$. Then G is a ranked space. Clearly, $(*)$ is fulfilled. It is easy to see that the conditions (1′), (2′), (3′) are satisfied, putting $\phi(\lambda, \mu=)\left[\frac{\lambda\mu}{2(\lambda+)\mu}\right]$, $\psi(\lambda)=\max.(\lambda-1, 0)$, and $\chi(\lambda\,;a)=\left[\frac{mx_1}{a_1+|a_2|}\right]$.

Since

$$av(m\,;e)=\left\{(a_1x_1, a_1x_2+a_2)\,;(x_1-1)^2+x_2^2<\frac{1}{m^2}\right\}$$

$$=\left\{(y_1, y_2)\,;(y_1-a_1)^2+(y_2-a_2)^2<\frac{a_1^2}{m^2}\right\},$$

$av(m\,;e)\subsetneqq v(n\,;a)$ for $n=\left[\frac{m}{a_1}\right]$, and $v(m\,;a)\subseteq av(n\,;e)$ for $n=[ma_1]$. Therefore (RG_4l) holds, and G is a ranked group.

Corresponding the element (a_1, a_2) of G to the point (a_1, a_2) in the complex plane, G is regarded as the half plane $a_1>0$, and $v(n\,;a)$ as the intersection of G and the circle of radius $\frac{1}{n}$ with center a. While, if we take $\{av(n\,;e)\}$ as $\mathfrak{B}_n(a)$, the neighbourhood of a with rank n is the circle with center a whose radius is $\frac{a_1}{n}$, and therefore depends upon a. Furthermore, if we take $\{v(n\,;e)a\}$ as $\mathfrak{B}_n(a)$, the neighbourhoods of $a(\neq e)$ are no longer circles.

References

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7) This is a well known example of a topological group in which the left and the right invariant Haar measures are essentialy different. See, for instance, [4], p. 256.