52. *On Generalized Integrals. II*

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In the preceding paper [5], we proposed a question whether the set of $(E.R.)$ integrable functions can be obtained as a completion of the set $\mathcal{E}$ with respect to some reasonable topology and rank ($\mathcal{E}$ stands for the set of step functions on $[a, b]$). The aim of a series of these papers is to give a positive answer to it. To do this, first of all in the Note I we introduced on $\mathcal{E}$ a topology and a rank so that $\mathcal{E}$ should become a ranked space. We proved that, when $u:\{V_n(f_n)\}$ is a fundamental sequence in $\mathcal{E}$, $f_n(x)$ converges to a finite function $f(x)$ a.e. and $\int_a^b f_n(x)dx$ converges to a finite limit, that is, every fundamental sequence u determines a function $J(u)=f(x)$ and a value $I(v)=\lim_{n\to\infty}\int_a^b f_n(x)dx$. Moreover, in this paper, we will establish that when we agree with two functions equal if they differ only in a set of measure zero, each maximal collection u^* in $\mathcal{E}$ determines a function which we can associate to this u^*. We denote this function by $J(u^*)$. Let us denote, by $\boldsymbol{K}$, the set of those functions $f(x)$ for which there exist fundamental sequences u with $J(u)=f(x)$, and denote, by $\boldsymbol{U}$, the set of all maximal collections. Then, *$J(u^*)$ is a (1,1) mapping of $\boldsymbol{U}$ onto $\boldsymbol{K}$* (Theorem 1). Furthermore, $\boldsymbol{K}$ coincides with the set of $(E.R.)$ integrable functions in the special sense (or A-integrable functions). It results from I, Corollary 2)[1] that for $u \in u^*$ and $v \in u^*$, we have $I(u)=I(v)$. Therefore, we can write this value $I=I(u^*)$. We take $I(f)=I(J^{-1}(f))$ as the value of the integral of $f(x)$ belonging to $\boldsymbol{K}$. Theorem 2 shows that $I(f)=(A)\int_a^b f(x)dx=(E.R.)\int_a^b f(x)dx$ for all $f\in \boldsymbol{K}$.

3. The mapping $J(u^*)$. Let us remark that in the ranked space $\mathcal{E}$ defined in the Note I, the *fundamental sequence* is defined in the following form: a monotone decreasing sequence of neighbourhoods $\{V_n(f_n);\ n=0, 1, 2, \cdots\}$ with $V_n(f_n)\in \mathfrak{V}_{\nu_n}$ is said to be fundamental if there exists a sub-sequence $\{V_{n_i}(f_{n_i});\ i=0,1,2,\cdots\}$ such that $f_{n_{2i}}=f_{n_{2i+1}}$ and $\nu_{n_{2i}}<\nu_{n_{2i+1}}$ (without the equality).

We continue the study of the fundamental sequence in $\mathcal{E}$. First, we show a few Lemmas.

1) The reference number indicates the number of the Note.

Lemma 5. *Given two measurable functions $f(x)$ and $g(x)$, for each $k>0$, we always have the relation*

$$\left|\int_a^b [f(x)+g(x)]^{2k}dx-\left(\int_a^b [f(x)]^k dx+\int_a^b [g(x)]^k dx\right)\right|\leq 2k \operatorname{mes}(E_1\cup E_2),$$

where $E_1=\{x\,;\,|f(x)|>k\}$ and $E_2=\{x\,;\,|g(x)|>k\}$.

Lemma 6. *If the neighbourhoods $V(A,\varepsilon\,;f)$ and $V(B,\eta\,;g)$ satisfy the following four conditions:*

(i) $|f(x)-g(x)|<\varepsilon-\eta$ *for all* $x\in A$,

(ii) $k \operatorname{mes}\{x\,;\,|f(x)-g(x)|>k\}<\varepsilon/3-\eta$ *for each* $k>0$,

(iii) $\left|\int_a^b [f(x)-g(x)]^k dx\right|<\varepsilon/3-\eta$ *for each* $k>0$,

(iv) $A\subseteq B$,

then $V(A,\varepsilon\,;f)\supseteq V(B,\eta\,;g)$.

Proof. Let $h\in V(g)$, and put $r(x)=h(x)-g(x)$ and $r_0(x)=g(x)-f(x)$. Then $h(x)$ can be written in the form $h(x)=f(x)+r_0(x)+r(x)$, and we have $h\in V(f)$. In fact, $[\alpha]$ and $[\beta]$ are easily seen. $[\gamma]$ results by using Lemma 5.

Lemma 7. *If $u:\{V(A_n,\varepsilon_n\,;f_n)\}$ is a monotone decreasing sequence such that $\lim_{n\to\infty}\varepsilon_n=0$ and $\operatorname{mes}([a,b]\backslash A_n)<\varepsilon_n$ for each n, there is a fundamental sequence $v:\{V(B_n,\eta_n\,;g_n)\}$ such that $v\succ u$ and $B_n\subseteq B_{n+1}$ for each n.*

Lemma 8. *If $u:\{V(A_n,\varepsilon_n\,;f_n)\}$ and $v:\{V(B_m,\eta_m\,;g_m)\}$ are two fundamental sequences such that*

$$\lim_{n\to\infty} f_n(x)=\lim_{m\to\infty} g_m(x) \qquad a.e.,$$

then there is a fundamental sequence w such that $w\succ u$ and $w\succ v$.

Proof. Without loss of generality, by Lemma 7 we can assume that $A_n\subseteq A_{n+1}$ and $B_m\subseteq B_{m+1}$. Let us choose, by induction, two index sequences $n_0<n_1<\cdots$ and $m_0<m_1<\cdots$ so that, for each i the conditions

$$\text{i)}\ \ n_i<m_i<n_{i+1}, \qquad \text{ii)}\ \ \varepsilon_{2n_i}\geq 4\eta_{2m_i},\ \eta_{2m_i}\geq 4\varepsilon_{2n_{i+1}}$$

can be satisfied. Put, for $i=0,1,2,\cdots$,

$$D_{2i}=A_{2n_i}\cap B_{2m_i},\ D_{2i+1}=B_{2m_i}\cap A_{2n_{i+1}};\ \kappa_{2i}=48\,\varepsilon_{2n_i},$$
$$\kappa_{2i+1}=48\,\eta_{2m_i};\ h_{2i}(x)=f_{2n_i}(x),\ h_{2i+1}(x)=g_{2m_i}(x),$$

and denote, by w^*, the sequence $\{V(D_i,\kappa_i\,;h_i)\}$. Then, for this, $V_{2i}(h_{2i})\supseteq V_{2i+1}(h_{2i+1})$ holds. For, if we put $f(x)=\lim_{n\to\infty} f_n(x)$, $r(x)=h_{2i+1}(x)-h_{2i}(x)$, $r_1(x)=h_{2i}(x)-f(x)$ and $r_2(x)=h_{2i+1}(x)-f(x)$, then, using by I, Lemma 3, (i): for every $x\in D_{2i}$, $|r(x)|\leq\varepsilon_{2n_i}+\eta_{2m_i}$, (ii): $k\operatorname{mes}\{x\,;\,|r(x)|>k\}\leq 2(\varepsilon_{2n_i}+\eta_{2m_i})$, (iii): $\left|\int_a^b [r(x)]^k dx\right|\leq\left|\int_a^b [r_1(x)]^{k/2}dx\right|+\left|\int_a^b [r_2(x)]^{k/2}dx\right|+k(\operatorname{mes}\{x\,;\,|r_1(x)|>k/2\}+\operatorname{mes}\{x\,;\,|r_2(x)|>k/2\})\leq 3(\varepsilon_{2n_i}+\eta_{2m_i})$. By definition, (iv): $D_i\subseteq D_{i+1}$. Hence, the asserted relation results from

Lemma 6, since $3(\varepsilon_{2n_i}+\eta_{2m_i})<\kappa_{2i}/3-\kappa_{2i+1}$. Similarly, we have $V_{2i+1}(h_{2i+1})\supseteq V_{2i+2}(h_{2i+2})$. Moreover, we have $\lim_{i\to\infty}\kappa_i=0$ and $\text{mes}([a, b]\backslash D_i)<\kappa_i$. Thus, there exists, by Lemma 7, a fundamental sequence w such that $w\succ w^*$. It follows easily that $V_{2i}(h_{2i})\supseteq V_{2n_i}(f_{2n_i})$, so that $w^*\succ u$. Similarly, we have $w^*\prec v$. This proves our assertion.

From now onwards, we don't distinguish between two functions which coincide almost everywhere. $\boldsymbol{K}$ denotes the set of those functions $f(x)$ for which there exist fundamental sequences u with $J(u)=f(x)$.

Proposition 3. *In order that a set u^* of fundamental sequences should be a maximal collection, it is necessary and sufficient that there exists a function $f(x)$ belonging to $\boldsymbol{K}$ and such that $u^*=\{u\,;J(u)=f(x)\}$.*

Proof. Necessity. Suppose that u^* is a maximal collection. Since then $u^*\neq\phi$, there is a $u\in u^*$. Put $J(u)=f(x)$, then for $v\in u^*$, there is a $w\in u^*$ such that $w\succ u$ and $w\succ v$, so that by I, Collollary 2 $J(v)=J(u)=f(x)$. Suppose, if possible, that there is a fundamental sequence w such that $w\notin u^*$ and $J(w)=f(x)$. Then, it follows from Lemma 8 that, by Zorn's Lemma, there exists a maximal collection containing u^* strictly, contrary to the property of maximal of u^*. Sufficiency. Let be $u_1\in u^*$ and $u_2\in u^*$, then, by Lemma 8 there is a fundamental sequence w such that $w\succ u_1$ and $w\succ u_2$, so that $J(w)=f(x)$. Therefore $w\in u^*$. Let v^* be a set of fundamental sequences containing u^* and with the property (1*). Since $u^*\neq\phi$, there is a $u\in u^*$. Therefore, for any $v\in u^*$, there is a $w\in v^*$ such that $w\succ u$ and $w\succ v$, so that $J(v)=J(u)=f(x)$ holds. Hence, we have $v\in u^*$, that is, $v^*\subseteq u^*$.

Therefore, each maximal collection u^* in $\mathcal{E}$ determines a function which we can associate to this maximal collection. We denote this function by $J(u^*)$. $\boldsymbol{U}$ denotes the set of all maximal collections u^*. Then, Proposition 3 asserts that:

Theorem 1. *The mapping $J(u^*)$ of $\boldsymbol{U}$ onto $\boldsymbol{K}$ is* (1, 1).

Since, by I, Corollary 2, we have $I(u)=I(v)$ for every $u\in u^*$ and $v\in u^*$, we can write $I=I(u^*)$ this value determined uniquely for u^*. Let us put, for $f\in\boldsymbol{K}$, $I(f)=I(J^{-1}(f))$. We will call this value the *integral* of $f(x)$.

4. The special (E.R.) integral and the A-integral. The method of the generalized integral, proposed by K. Kunugi in [3] and called the $(E.R.)$ integral, admits the investigation in abstract measure spaces. The case of locally compact topological group provided with a Haar measure was discussed completely by T. Ikegami in [2]. On the other hand, K. Kunugi remarked in [3] that the method of change of the variable admits the extension of the range of the integration, and presented in [4] the precise definition of the integral. He called this integral the $(E.R.)$ integral of Stieltjes type with respect to a function $g(x)$, and denoted by $(E.R.)\int_a^b f(x)dg$ the value of this integral. The definition of the integral of this type in abstract measure spaces can be found in [6]. H. Okano called it the $(E.R.\ \nu)$ integral with respect to a measure ν. Hence, from now onwards, we will call the integral, introduced firstly by K. Kunugi, the *special* $(E.R.)$ *integral.*

A function $f(x)$ is said to be A-integrable on $[a, b]$, if

$$\text{mes}\{x;|f(x)|>n\}=o(1/n),$$

and if the limit of the integrals of truncated functions

$$\lim_{n\to\infty}\int_a^b [f(y)]^n dx$$

exists. The value of the limit is said to be the *A-integral* of $f(x)$ on $[a, b]$. Titchmarsh [7] introduced the notion of this integral and called it the Q integral.

As it is already proved by I. Amemiya and T. Ando [1], the special $(E.R.)$ integral is equivalent to the A-integral, in such a sense that if one exists, then so does the other and their values are equal.

Lemma 9. *If* $u:\{V(A_n, \varepsilon_n; f_n)\}$ *is a fundamental sequence, then we have*

1) $\lim_{k\to\infty} k\ \text{mes}\{x;|J(u)|>k\}=0,$

2) $I(u)=\lim_{k\to\infty}\int_a^b [J(u)]^k dx,$

where k runs through the set of all positive numbers.

Proof. Put $\gamma_n=\sup_x|f_n(x)|$. Then, for very sufficiently great positive number k, if n is a positive integer for which $k/2\geq\gamma_n$ holds, we have by I, Lemma 3 $k\ \text{mes}\{x;|J(u)|>k\}\leq 2(k/2\ \text{mes}\{x;|J(u)-f_n(x)|>k/2\}+k/2\ \text{mes}\{x;|f_n(x)|>k/2\})<2\varepsilon_n$, and so we obtain 1). By Lemma 5, $\left|\int_a^b [J(u)]^k dx-\int_a^b f_n(x)dx\right|\leq 2k\ \text{mes}\{x;|J(u)|>\text{k}\}+\left|\int_a^b [J(u)-f_n(x)]^{2k}dx\right|\leq 5\varepsilon_n$. Hence, 2) results from I, Corollary 1.

Lemma 10. *If $f(x)$ is the A-integrable function, then there exists a fundamental sequence* $u:\{V_n(f_n)\}$ *such that* $\lim_{n\to\infty} f_n(x)=f(x)$.

Proof. Let us put, for $k>0$,

$$\eta_k=\left|(A)\int_a^b f(x)dx-\int_a^b [f(x)]^k dx\right|,$$

$$\lambda_k=k\ \text{mes}\{x;|f(x)|>k\}.$$

As $f(x)$ is A-integrable, we have $\lim_{k\to\infty}\eta_k=0$ and $\lim_{k\to\infty}\lambda_k=0$. Therefore, there is an increasing sequence n_i $(i=1, 2, \cdots)$ of positive integers such that, if we put $\varepsilon_i=\max(\max_{k\geq n_i}\lambda_k, \max_{k\geq n_i}\eta_k, 1/2^{4i})$, we have $\varepsilon_i\geq 4\varepsilon_{i+1}$. Put $f_i(x)=[f(x)]^{n_i}$ and $A_i=\{x;|f(x)|\leq n_i\}$. Then, as it is easily seen, there exist a sequence $\{g_i(x)\}$ of step functions and a monotone increasing sequence $\{B_i\}$ of closed sets, which have the following properties:

1) $n_i\geq|g_i(x)|$,
2) $|f_i(x)-g_i(x)|<\varepsilon_i/b-a$ for all $x\in B_i$,
3) $B_i\subseteq A_i$ and $\mathrm{mes}(A_i\backslash B_i)<\varepsilon_i/n_i$.

Let us put $\kappa_i=2^7\varepsilon_i$, then the sequence $v:\{V(B_i, \kappa_i; g_i)\}$ is a monotone decreasing sequence. In fact, if we put $r(x)=g_{i+1}(x)-g_i(x)$, then (i) $|r(x)|<\kappa_i-\kappa_{i+1}$ for all $x\in B_i$. (ii): $k\,\mathrm{mes}\{x;|r(x)|>k\}<\kappa_i/3-\kappa_{i+1}$ holds. Because, we have, for any $k>0$, $k\,\mathrm{mes}\{x;|g_i(x)-f_i(x)|>k\}\leq k\,\mathrm{mes}\{x;|g_i(x)-f_i(x)|>k, x\in B_i\}+2n_i(\mathrm{mes}([\mathrm{a,b}]\backslash A_i)+\mathrm{mes}(A_i\backslash B_i))<5\varepsilon_i$, and $k\,\mathrm{mes}\{x;|f_{i+1}(x)-f_i(x)|>k\}\leq k\,\mathrm{mes}\{x;|f(x)|>k+n_i\}\leq\varepsilon_i$. (iii): $\int_a^b|[g_{i+1}(x)-g_i(x)]^k-[f_{i+1}(x)-f_i(x)]^k|dx\leq\int_a^b|g_{i+1}(x)-f_{i+1}(x)|dx+\int_a^b|g_i(x)-f_i(x)|dx\leq\varepsilon_{i+1}+2n_{i+1}\mathrm{mes}([a,b]\backslash B_{i+1})+\varepsilon_i+2n_i\,\mathrm{mes}([a,b]\backslash B_i)\leq 5(\varepsilon_{i+1}+\varepsilon_i)$. Moreover, for any $k>0$, $\left|\int_a^b[f_i(x)-f(x)]^k dx\right|\leq\left|\int_a^b[(f(x))^{n_i+k}-f(x)]dx\right|+\left|\int_a^b([f(x)]^{n_i}-f(x))dx\right|\leq 2\varepsilon_i$. From Lemma 5, $\left|\int_a^b[f_{i+1}(x)-f_i(x)]^k dx\right|\leq\left|\int_a^b[f_{i+1}(x)-f(x)]^{k/2}dx\right|+\left|\int_a^b[f_i(x)-f(x)]^{k/2}dx\right|+k\,\mathrm{mes}(E_1\cup E_2)$, where $E_j=\{x;|f_{i+j}(x)-f(x)|>k/2\}$ for $j=0,1$. Further we have $k\,\mathrm{mes}(E_1\cup E_2)\leq 2(\varepsilon_i+\varepsilon_{i+1})$, since $E_j=\{x;|f(x)|>k/2+n_{i+j}\}$. Thus $\left|\int_a^b[r(x)]^k dx\right|<\kappa_i/3-\kappa_{i+1}$ results. (iv): From definition, $B_i\subseteq B_{i+1}$. Therefore, applying Lemma 6, $V_i(g_i)\supseteq V_{i+1}(g_{i+1})$ holds. Moreover, we have $\lim_{i\to\infty}\kappa_i=0$, $\mathrm{mes}([a,b]\backslash B_i)=0$ and $\lim_{i\to\infty}g_i(x)=f(x)$. Hence, our assertion holds, paying attention to Lemma 7 and I, Lemma 4.

We now have the following;

Theorem 2. *K coincides with the set of all A-integrable functions (or (E.R.) integrable functions in the special sense), and we have*

$$I(f)=(A)\int_a^b f(x)dx=(E.R.)\int_a^b f(x)dx.$$

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