50. A Characterization of Haar Subspaces in C[a, b]*)

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Let M be an n-dimensional subspace of the space $C[a, b]$ with Tchebycheff norm:

$$||f||=\max\{|f(x)|: a\leq x\leq b\}$$

It is well known that the following conditions are mutually equivalent [1]:

(A) If a pair of functions in M agrees on any set of n distinct points in $[a, b]$ then they agree on the entire interval $[a, b]$;

(B) For any basis $\{g_1, \cdots, g_n\}$ of M and for any set of n distinct points $x_1, \cdots, x_n$ in $[a, b]$, the determinant $\det(g_i(x_j))$ is different from 0;

(C) Each element f in $C[a, b]$ has a unique best approximation in M (with respect to the Tchebycheff norm).

Any n-dimensional subspace M of $C[a, b]$ satisfying one of the above conditions (A)—(C) is known as a *Haar* subspace. The purpose of this paper is to show that each one of the above conditions is further equivalent to the following condition:

(D) For each f in $C[a, b]$ which is not identically zero on $[a, b]$ and for each best approximation p in M to f, the following inequality is valid:

$$||p||<2||f||$$

(C)$\Rightarrow$(D). Suppose that (C) is true and let p be a best approximation in M to a non-zero function f in $C[a,b]$. We may assume that $p\neq 0$. Then, from uniqueness,

$$||p-f||<||0-f||$$

and therefore,

$$||p||\leq||p-f||+||f||<||0-f||+||f||=2||f||$$

(D)$\Rightarrow$(B). Suppose that (B) is false. We must show that there exists a nonzero function f in $C[a, b]$ and a best approximation p in M to f

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such that $\|p\| \geq 2\|f\|$ (actually, the inequality $\|p\|>2\|f\|$ is impossible since $\|p-f\|<\|0-f\|$). Since (B) is false, there exists a set of n distinct points, say $\{x_1, \cdots, x_n\}$, such that the determinant $\det(g_i(x_j))$ vanishes, where $\{g_1, \cdots, g_n\}$ is some basis for M. This means that the rows and columns of the determinant are linearly dependent. For each x in $[a, b]$, let $\hat{x}$ be the n-vector $[g_1(x), \cdots, g_n(x)]$. Then, from the row dependence, there exists a sets of real numbers, $\{c_1, \cdots, c_n\}$, such that

$$0=\sum_{i=1}^{n} c_i\hat{x}_i=\sum_{i=1}^{n}|c_i|(\operatorname{sgn} c_i)\hat{x}_i \tag{1}$$

and

$$\sum_{i=1}^{n}|c_i|=1.$$

Similarly, from the column dependence,

$$0=\sum_{i=1}^{n} d_i g_i(x_j), \qquad j=1, \cdots, n \tag{2}$$

for a set of real numbers $\{d_1, \cdots, d_n\}$ where $\sum_{i=1}^{n}|d_i|>0$. Set

$$p=\sum_{i=1}^{n} d_i g_i. \tag{3}$$

Then, p is in M and $p\neq 0$. We now assume that the constants $d_1, \cdots, d_n$ are so chosen that we have $\|p\|=1$. We will construct a function f in $C[a, b]$ such that $\|f\|=1$ and $2p$ is a best approximation in M to this f. This will complete the proof.

From the fact that $\|p\|=1$, we have pointwise

$$1\geq \min\{2p+1, 1\}\geq \max\{2p-1, -1\}\geq -1.$$

Choose a continuous function e on $[a, b]$ such that

$$1\geq \min\{2p+1, 1\}\geq e\geq \max\{2p-1, -1\}\geq -1. \tag{4}$$

But because p vanishes at x_j, $j=1, \cdots, n$, we have

$$\min\{2p(x_j)+1, 1\}=1 \quad \text{and} \quad \max\{2p(x_j)-1, -1\}=-1$$

$j=1, \cdots, n$. Hence, we may impose on e condition

$$e(x_j)=\operatorname{sgn} x_j=1 \text{ or } -1 \qquad j=1, \cdots, n \tag{5}$$

without disturbing condition (4). Now set $f=2p-e$. Then $\|f\|=\|e\|=1$. Furthermore, because of (1) and (5), the function 0 is in the convex hull of the set $\{e(x)\hat{x} : |e(x)|=\|e\|=1\}$. This shows that $2p$ is a best approximation in M to f [1].

Reference

[1] E. W. Cheney: Introduction to Approximation Theory. Mc-Graw Hill, New York (1966).