49. *Calculus in Ranked Vector Spaces. II*

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1.6. Ranked vector space. In what follows we denote by $\mathfrak{R}$ the space of all real numbers with the usual topology.

(1.6.1) **Definition.** A space E which is satisfying the following Conditions (I), (II) is called a *ranked vector space.*

(1.6.2) (I) E is a vector space over the real or complex numbers and there is a countably family $\mathfrak{V}_0(0), \mathfrak{V}_1(0), \mathfrak{V}_2(0), \cdots, \mathfrak{V}_n(0), \cdots$ where each $\mathfrak{V}_n(0)$ consists of subsets of E. Let $\mathfrak{V}(0)=\cup\mathfrak{V}_n(0)$, then it satisfies the following conditions:

(A) Every V belonging to $\mathfrak{V}(0)$ contains zero;

(B) For any U, $V\in\mathfrak{V}(0)$, there exists a $W\in\mathfrak{V}(0)$ such that
$$W\subset U\cap V;$$

(a) For any $U\in\mathfrak{V}(0)$, and for an integer n $(0\leq n<\omega_0)$, there exists an integer m and a $V\in\mathfrak{V}(0)$ such that
$$m\geq n, \quad V\in\mathfrak{V}_m(0), \quad \text{and} \quad V\subset U;$$

(b) $E\in\mathfrak{V}_0(0)$.

With each element $x\in E$ there is associated a non-empty set $\mathfrak{V}(x)$ as follows:
$$\mathfrak{V}(x)=\{x+V;\ V\in\mathfrak{V}(0)\}.$$
Every element $U=x+V\in\mathfrak{V}(x)$ is called a *neighborhood* of a point x. Further, there is a countably system $\{\mathfrak{V}_n\}$ defined by
$$\mathfrak{V}_n=\{x+V;\ x\in E,\ V\in\mathfrak{V}_n(0)\},$$
for $n=0, 1, 2, \cdots$.

(1.6.3) (II) In E the following axioms hold [1]:

(1) *There exists a non-negative function $\phi(\lambda,\mu)$, defined for $\lambda\geq 0$ and $\mu\geq 0$, such that $\lim_{\lambda,\mu\to\infty}\phi(\lambda,\mu)=\infty$, and the following holds: if $U\in\mathfrak{V}_l(0)$, $V\in\mathfrak{V}_m(0)$, $W\in\mathfrak{V}_n(0)$, $n\leq\phi(l,m)$, and $U+V\subset W$, then there is an integer $n^*\geq\phi(l,m)$, and a neighborhood $W^*\in\mathfrak{V}_{n^*}(0)$, such that*
$$U+V\subset W^*\subset W.$$

(2) *There exists a non-negative function $\psi(\lambda,\mu)$ defined for $\lambda\geq 0$ and $\mu\geq 1$ such that $\lim_{\lambda\to\infty}\psi(\lambda,\mu)=\infty$, for each fixed μ, and the following holds: let α be a scalar with $|\alpha|\geq 1$. If $U\in\mathfrak{V}_m(0)$, $V\in\mathfrak{V}_n(0)$, $\alpha U\subset V$, and $n\leq\psi(m,|\alpha|)$, then there is an integer $n^*\geq\psi(m,|\alpha|)$ and a $V\in\mathfrak{V}_{n^*}(0)$ such that*
$$\alpha U\subset V^*\subset V.$$

(3) *Let* $U \in \mathfrak{B}(0)$ *and* $x \in U$. *Then for any integer* n, *there is an integer* $m \geq n$, *a neighborhood* $V \in \mathfrak{B}_m(0)$ *and some positive* ρ *such that*

$$\rho x \in V \subset U.$$

Moreover, we assume that *every* V *in* $\mathfrak{B}(0)$ *is circled* (i.e., for any $x \in V$ and for any α with $|\alpha| \leq 1$, $\alpha x \in V$).

Then it is clear that E is a ranked space with the indicator ω_0.

If a space E satisfies the following condition:

(1.6.4) *For any* $U \in \mathfrak{B}_l(0)$ *and* $V \in \mathfrak{B}_m(0)$, *there exists an integer* n *such that*

$$n \geq \max(l, m) \quad \text{and} \quad U \cap V \in \mathfrak{B}_n(0),$$

then we can replace Axioms (1), (2), (3) of (1.6.3) by the following axioms:

(1.6.5) (1′) *There exists a function* $\phi(\lambda, \mu)$ *such as* ϕ *in* (1) *of* (1.6.3), *and the following holds; for any* $U \in \mathfrak{B}_l(0)$ *and for* $V \in \mathfrak{B}_m(0)$, *there is an integer* n *and a neighborhood* W *such that*

$$n \geq \phi(l, m), \quad W \in \mathfrak{B}_n(0), \quad \text{and} \quad U + V \subset W.$$

(2′) *There exists a function* $\psi(\lambda, \mu)$ *such as* ψ *in* (2) *of* (1.6.3), *the following holds; for any* $U \in \mathfrak{B}_m(0)$ *and for* α *with* $|\alpha| \geq 1$, *there is an integer* n *and a neighborhood* V *such that*

$$n \geq \psi(m, |\alpha|), \quad V \in \mathfrak{B}_n(0), \quad \text{and} \quad \alpha U \subset V.$$

(3′) *For any integer* n *and for any* x *in* E, *there is an integer* m, *a neighborhood* V *and a positive number* ρ *such that*

$$m \geq n, \quad V \in \mathfrak{B}_m(0), \quad \text{and} \quad \rho x \in V.$$

(1.6.6) **Example.** Let E be a normed vector space, then it is a ranked vector space.

In fact, we define $\{\mathfrak{B}_n(0)\}$ as follows:

$$\mathfrak{B}_n(0) = \{V_n(0)\}, \qquad \text{for } n = 0, 1, 2, \cdots$$

where $V_n(0) = \left\{x \,;\, ||x|| < \dfrac{1}{n}\right\}$, for $n = 1, 2, 3, \cdots$, and $V_0(0) = E$. Then we have

(1.6.7) $$m \geq n \Rightarrow V_m(0) \subset V_n(0).$$

Indeed, let $x \in V_m(0)$, then $||x|| < \dfrac{1}{m}$. Since $m \geq n$, $\dfrac{1}{m} \leq \dfrac{1}{n}$.

$$\therefore \quad ||x|| < \frac{1}{n} \qquad \therefore \quad x \in V_n(0).$$

Let

$$\mathfrak{B}(0) = \{V_0(0), V_1(0), V_2(0), \cdots, V_n(0), \cdots\},$$

for any $x \in E$

$$\mathfrak{B}(x) = \{x + V_n(0) \,;\, n = 0, 1, 2, \cdots.\}$$

and

$$\mathfrak{B}_n = \{x + V_n(0) \,;\, x \in E\}, \qquad \text{for } n = 0, 1, 2, \cdots.$$

It is clear that Condition (1.6.2) holds.

Let us show that Condition (1.6.3) holds. Since in this case by (1.6.7) Condition (1.6.4) holds, it suffices to show that Axioms (1′), (2′), (3′) hold.

Let $\phi(\lambda, \mu)=\min\left(\left[\frac{\lambda}{2}\right], \left[\frac{\mu}{2}\right]\right)$. If $U\in\mathfrak{V}_l(0)$ and $V\in\mathfrak{V}_m(0)$, i.e., $U=V_l(0)$ and $V=V_m(0)$, then for any $x\in U$, $y\in V$, $\|x\|<\frac{1}{l}$, $\|y\|<\frac{1}{m}$. Here we may assume that $l\geq m$. Then since $\phi(l, m)=\left[\frac{m}{2}\right]$, if we put $n=\left[\frac{m}{2}\right]$, then $n\leq\frac{m}{2}$ $\quad\therefore\quad \frac{1}{n}\geq\frac{2}{m}$.

$$\therefore\quad \|x+y\|\leq\|x\|+\|y\|<\frac{1}{l}+\frac{1}{m}\leq\frac{2}{m}\leq\frac{1}{n}.$$

$$\therefore\quad x+y\in V_n(0). \qquad \therefore\quad U+V\subset V_n(0).$$

where $n\geq\phi(l, m)$. Therefore Axiom (1′) holds.

Let $\psi(\lambda, \mu)=\left[\frac{\lambda}{\mu}\right]$. For any $U\in\mathfrak{V}_m(0)$ and for a scalar α with $|\alpha|\geq1$, let $x\in U$, then we have $\|\alpha x\|=|\alpha|\,\|x\|<|\alpha|\frac{1}{m}$.

Since $\psi(m, |\alpha|)=\left[\frac{m}{|\alpha|}\right]$, if we put $n=\frac{m}{|\alpha|}$, then $\frac{|\alpha|}{m}\leq\frac{1}{n}$.

$$\therefore\quad \|\alpha x\|<\frac{1}{n}, \qquad \therefore\quad \alpha x\in V_n(0).$$

$$\therefore\quad \alpha U\subset V_n(0),$$

where $n\geq\psi(m, |\alpha|)$. Therefore Axiom (2′) holds.

Let $x\in E$ and n arbitrary non-negative integer. Put $m=n+1$, then there exists a positive number ρ such that

$$\rho\|x\|<\frac{1}{m}, \qquad \therefore\quad \rho x\in V_m(0)\in\mathfrak{V}_m(0).$$

Thus a normed vector space E is a ranked vector space.

In a ranked vector space E the following proposition holds:

(1.6.8) **Proposition.** *If E is a ranked vector space and $\{x_n\}$ is a sequence in E, then $\{\lim x_n\}\ni x$ is equivalent to $\{\lim(x_n-x)\}\ni 0$.*

Proof. It follows from the definition of $\{\lim x_n\}\ni x$ that there exists a sequence $\{U_n(x)\}$ of neighborhoods of x and a sequence $\{\alpha_n\}$ of integers such that

$$U_0(x)\supset U_1(x)\supset U_2(x)\supset\cdots\supset U_n(x)\supset\cdots,\ 0\leq n<\omega_0,$$
$$\alpha_0\leq\alpha_1\leq\alpha_2\leq\cdots\leq\alpha_n\leq\cdots,\ 0\leq n<\omega_0,$$
$$\sup_n \alpha_n=\omega_0, \quad U_n(x)\ni x_n, \quad\text{and}\quad U_n(x)\in\mathfrak{V}_{\alpha_n},$$

for $n=0, 1, 2, \cdots$.

Since by (1.6.2) $U_n(x)$ can be written in the form $x+V_n(0)$, $V_n(0)\in\mathfrak{V}_{\alpha_n}$, using $U_n(x) \ni x_n$, we have $V_n(0) \ni x_n - x$, where $V_n(0) \in \mathfrak{V}_{\alpha_n}$. Therefore

$$\{\lim (x_n - x)\} \ni 0.$$

It is obvious that the converse assertion holds.

Since the following Propositions (1.6.9), (1.6.10), (1.6.11) are proved in the paper of M. Washihara [2], we shall omit.

(1.6.9) **Proposition.** *Let E be a ranked vector space, $\{x_n\}$, $\{y_n\}$ sequences in E, and $x, y \in E$. If $\{\lim x_n\} \ni x$ and $\{\lim y_n\} \ni y$, then*

$$\{\lim (x_n + y_n)\} \ni x + y.$$

This means the continuity of addition.

(1.6.10) **Proposition.** *Let E be a ranked vector space, $\{x_n\}$ a sequence in E and $x \in E$. If $\{\lim x_n\} \ni x$, then, for any $\lambda \in \mathfrak{R}$,*

$$\{\lim \lambda x_n\} \ni \lambda x.$$

(1.6.11) **Proposition.** *Let E be a ranked vector space and $x \in E$. If $\lim \lambda_n = \lambda$ in $\mathfrak{R}$, then*

$$\{\lim \lambda_n x\} \ni \lambda x.$$

(1.6.12) **Proposition.** *Let E be a ranked vector space, $\{x_n\}$ a sequence in E and $x \in E$. If $\{\lim x_n\} \ni x$ in E and $\lim \lambda_n = \lambda$ in $\mathfrak{R}$, then*

$$\{\lim \lambda_n x_n\} \ni \lambda x.$$

Proof. (a) We shall consider the following special case:

$$\lambda = 0, \qquad x = 0.$$

Then it follows from the definition of $\{\lim x_n\} \ni 0$ that there exists a sequence $\{U_n(0)\}$ of neighborhoods of zero and a sequence $\{\alpha_n\}$ of integers such that

$$U_0(0) \supset U_1(0) \supset U_2(0) \supset \cdots \supset U_n(0) \supset \cdots, \quad 0 \leq n < \omega_0,$$
$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \cdots, \quad 0 \leq n < \omega_0,$$
$$\sup_n \alpha_n = \omega_0, \quad U_n(0) \ni x_n, \quad \text{and} \quad U_n(0) \in \mathfrak{V}_{\alpha_n},$$

for $n = 0, 1, 2, \cdots$.

Since $\lim \lambda_n = 0$, there is an positive integer N such that

$$n \geq N \Rightarrow |\lambda_n| \leq 1.$$

It follows, using the assumption that each $U_n(0)$ is circled, that

$$U_N(0) \ni \lambda_N x_N, \ U_{N+1}(0) \ni \lambda_{N+1} x_{N+1}, \cdots$$
$$\therefore \quad \{\lim_n \lambda_{N+n} x_{N+n}\} \ni 0.$$

By (1. 2. 3), we have

$$\{\lim \lambda_n x_n\} \ni 0.$$

(b) Let $\{\lim x_n\} \ni x$ and $\lim \lambda_n = \lambda$, i.e., $\{\lim (x_n - x)\} \ni 0$ and $\lim (\lambda_n - \lambda) = 0$. From (a), we have

$$\{\lim (\lambda_n - \lambda)(x_n - x)\} \ni 0.$$
$$\therefore \quad \{\lim (\lambda_n x_n - \lambda x_n - \lambda_n x + \lambda x)\} \ni 0.$$

By (1.6.9), (1.6.10), (1.6.11) the following hold,

$$\{\lim (\lambda x_n + \lambda_n x - \lambda x)\} \ni \lambda x + \lambda x - \lambda x = \lambda x.$$
$$\therefore \quad \{\lim \lambda_n x_n\} \ni \lambda x.$$

(1.6.13) **Proposition.** *Let $E_1, E_2, \cdots, E_m$ be a family of ranked* *vector spaces and let* $\{z_n\}=\{(x_{n1}, x_{n2}, \cdots, x_{nm})\}$, $\{z'_n\}=\{(x'_{n1}, x'_{n2}, \cdots, x'_{nm})\}$ *be sequences in the direct product* $\times E_i$, *and* $z=(x_1, x_2, \cdots, x_m)$, $z'=(x'_1, x'_2, \cdots, x'_m) \in \times E_i$. *If* $\{\lim z_n\} \ni z$ *and* $\{\lim z'_n\} \ni z'$, *then*

$$\{\lim (z_n + z'_n)\} \ni z + z'.$$

Proof. By (1.5.1), $\{\lim z_n\} \ni z$ and $\{\lim z'_n\} \ni z'$ are equivalent to the following:

$$\{\lim x_{n1}\} \ni x_1, \{\lim x_{n2}\} \ni x_2, \cdots, \{\lim x_{nm}\} \ni x_m,$$

and

$$\{\lim x'_{n1}\} \ni x'_1, \{\lim x'_{n2}\} \ni x'_2, \cdots, \{\lim x'_{nm}\} \ni x'_m.$$

Since $E_1, E_2, \cdots, E_m$ are ranked vector spaces, by (1.6.9) we have

$$\{\lim (x_{n1}+x'_{n1})\} \ni x_1+x'_1, \{\lim (x_{n2}+x'_{n2})\} \ni x_2+x'_2, \cdots,$$
$$\{\lim (x_{nm}+x'_{nm})\} \ni x_m+x'_m.$$
$$\therefore \quad \{\lim (z_n+z'_n)\} \ni z+z'.$$

(1.6.14) **Proposition.** *Let* $E_1, E_2, \cdots, E_m$ *be a family of ranked vector spaces and let* $\{z_n\}=\{(x_{n1}, x_{n2}, \cdots, x_{nm})\}$ *be a sequence in the direct product* $\times E_i$, *and* $z=(x_1, x_2, \cdots, x_m) \in \times E_i$. *If* $\{\lim z_n\} \ni z$, *then, for any* $\lambda \in \mathfrak{R}$,

$$\{\lim \lambda z_n\} \ni \lambda z.$$

Proof. $\{\lim z_n\} \ni z$ is equivalent to the following:

$$\{\lim x_{n1}\} \ni x_1, \{\lim x_{n2}\} \ni x_2, \cdots, \{\lim x_{nm}\} \ni x_m.$$

Since $E_1, E_2, \cdots, E_m$ are ranked vector spaces, by (1.6.10) we have for any $\lambda \in \mathfrak{R}$,

$$\{\lim \lambda x_{n1}\} \ni \lambda x_1, \{\lim \lambda x_{n2}\} \ni \lambda x_2, \cdots, \{\lim \lambda x_{nm}\} \ni \lambda x_m.$$
$$\therefore \quad \{\lim \lambda z_n\} \ni \lambda z.$$

(1.6.15) **Proposition.** *Let* $z=(x_1, x_2, \cdots, x_m)$ *be an arbitrary element of the direct product* $\times E_i$ *of the ranked spaces* $E_1, E_2, \cdots, E_m$. *If* $\lim \lambda_n = \lambda$ *in* $\mathfrak{R}$, *then*

$$\{\lim \lambda_n z\} \ni \lambda z.$$

Proof. If $\lim \lambda_n = \lambda$ in $\mathfrak{R}$, by (1.6.11), we get

$$\{\lim \lambda_n x_1\} \ni \lambda x_1, \{\lim \lambda_n x_2\} \ni \lambda x_2, \cdots, \{\lim \lambda_n x_m\} \ni \lambda x_m.$$
$$\therefore \quad \{\lim \lambda_n z\} \ni \lambda z.$$

(1.6.16) **Proposition.** *Let* $E_1, E_2, \cdots, E_m$ *be a family of ranked vector spaces and let* $\{z_n\}=\{(x_{n1}, x_{n2}, \cdots, x_{nm})$ *a sequence of the direct product* $\times E_i$ *and* $z=(x_1, x_2, \cdots, x_m) \in \times E_i$. *If* $\{\lim z_n\} \ni z$ *in* $\times E_i$ *and* $\lim \lambda_n = \lambda$ *in* $\mathfrak{R}$, *then*

$$\{\lim \lambda_n z_n\} \ni \lambda z.$$

Proof. By (1.5.1) $\{\lim z_n\} \ni z$ is equivalent to the following:

$$\{\lim x_{n1}\} \ni x_1, \{\lim x_{n2}\} \ni x_2, \cdots, \{\lim x_{nm}\} \ni x_m.$$

It follows from (1.6.12), using the assumption $\lim \lambda_n = \lambda$, that

$$\{\lim \lambda_n x_{n1}\} \ni \lambda x_1, \{\lim \lambda_n x_{n2}\} \ni \lambda x_2, \cdots, \{\lim \lambda_n x_{nm}\} \ni \lambda x_m.$$
$$\therefore \quad \{\lim \lambda_n z_n\} \ni \lambda z.$$

In the direct product $\times E_i$ of the ranked vector spaces $E_1, E_2, \cdots,$

E_m (1.6.13) means the continuity of addition, and (1.6.14), (1.6.15), (1.6.16) mean the continuity of the scalar multiplication. Thus we may consider the direct product $\times E_i$ of the ranked spaces $E_1, E_2, \cdots, E_m$ as a ranked vector space.

1.7. Quasi-bounded sequence. (1.7.1) **Definition.** Let $\{x_n\}$ be a sequence in a ranked vector space E. Then a sequence $\{x_n\}$ is called a *quasi-bounded sequence* in E if and only if, for any sequence $\{\mu_n\}$ in $\mathfrak{R}$ with $\mu_n \to 0$,

$$\{\lim \mu_n x_n\} \ni 0.$$

(1.7.2) **Proposition.** *Let $\{x_n\}$ be a sequence in a ranked vector space E. If $\{\lim x_n\} \ni x$ in E, then $\{x_n\}$ is a quasi-bounded sequence.*

Proof. Let $\{\mu_n\}$ be an arbitrary sequence in $\mathfrak{R}$ with $\mu_n \to 0$, then it follows from (1.6.12), using the assumption that $\{\lim x_n\} \ni x$, that

$$\{\lim \mu_n x_n\} \ni 0 \cdot x = 0.$$

Hence $\{x_n\}$ is a quasi-bounded sequence.

(1.7.3) In particular, if $x_n = x$ for $n = 0, 1, 2, \cdots$, then by (1.2.4) $\{\lim x_n\} \ni x$. Therefore it is also a quasi-bounded sequence.

One easily verifies the following proposition:

(1.7.4) **Proposition.** *Let $\{x_n\}$ be a quasi-bounded sequence in a ranked vector space E. If $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, then it is also a quasi-bounded sequence.*

(1.7.5) **Proposition.** *Let $\{x_n\}$, $\{y_n\}$ be two quasi-bounded sequences in a ranked vector space E, and α, β, γ arbitrary numbers of $\mathfrak{R}$. Then $\{\alpha x_n\}$, and $\{\beta x_n + \gamma y_n\}$ are also quasi-bounded sequences.*

Proof. It is suffices to show that $\{\beta x_n + \gamma y_n\}$ is a quasi-bounded sequence. Let $\{\mu_n\}$ be a sequence in $\mathfrak{R}$ with $\mu_n \to 0$.

$$\mu_n(\beta x_n + \gamma y_n) = (\beta \mu_n) x_n + (\gamma \mu_n) y_n.$$

By assumption we have

$$\{\lim (\beta \mu_n) x_n\} \ni 0, \quad \text{and} \quad \{\lim (\gamma \mu_n) y_n\} \ni 0.$$

$$\therefore \quad \{\lim \mu_n(\beta x_n + \gamma y_n)\} \ni 0.$$

Thus if $\{x_n\}$, $\{y_n\}$ are quasi-bounded sequences, then, for any $\alpha, \beta, \gamma \in \mathfrak{R}$,

$$\{\alpha x_n\}, \quad \{\beta x_n + \gamma y_n\}$$

are also quasi-bounded sequences.

References

[1] M. Washihara: On ranked spaces and linearity. Proc. Japan Acad., **43**, 584–589 (1967).

[2] ——: loc. cit.