48. *Calculus in Ranked Vector Spaces. I*

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§ 1. Ranked vector space. 1.1. Ranked space. Let E be a neighborhood space, i.e., with every element $x \in E$ there is associated a non-empty set $\mathfrak{V}(x)=\{V(x)\}$ of subsets of E such that

(1.1.1) (1) $V(x) \in \mathfrak{V}(x) \Rightarrow V(x) \ni x$;

(2) For any $U(x)$, $V(x) \in \mathfrak{V}(x)$, there exists a $W(x) \in \mathfrak{V}(x)$ such that

$$W(x) \subset U(x) \cap V(x);$$

(3) $$E \in \mathfrak{V}(x).$$

Every element $V(x)$ of $\mathfrak{V}(x)$ is called a *neighborhood* of a point $x \in E$.

(1.1.2) **Definition.** A neighborhood space E, on which a countably system $\mathfrak{V}_0, \mathfrak{V}_1, \mathfrak{V}_2, \cdots, \mathfrak{V}_n, \cdots$ consisting of neighborhoods ($E \in \mathfrak{V}_0$) is defined, is called a ranked space with the indicator ω_0 if and only if for every $x \in E$, $U(x) \in \mathfrak{V}(x)$ and for an integer n ($0 \leq n < \omega_0$) there exists an integer m ($0 \leq m < \omega_0$) and a neighborhood $V(x) \in \mathfrak{V}(x)$ such that

$$m \geq n, \ V(x) \in \mathfrak{V}_m \text{ and } V(x) \subset U(x).$$

A metric space is a ranked space. Another examples of ranked spaces shall be found in the paper of K. Kunugi [1].

1.2. Convergence. Let $\{x_n\}$ be a sequence in a ranked space E. Now we shall consider a convergence introduced by K. Kunugi [2].

(1.2.1) **Definition.** We say that *a sequence* $\{x_n\}$ *converges to* x in a ranked space E, and we write $\{\lim_n x_n\} \ni x$ if and only if there exists a sequence $\{V_n(x)\}$ of neighborhoods and a sequence $\{\alpha_n\}$ of integers such that

$$V_0(x) \supset V_1(x) \supset V_2(x) \supset \cdots \supset V_n(x) \supset \cdots, \ 0 \leq n < \omega_0,$$
$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \cdots, \ 0 \leq n < \omega_0,$$
$$\sup_n \alpha_n = \omega_0, \ V_n(x) \ni x_n, \text{ and } V_n(x) \in \mathfrak{V}_{\alpha_n}(x),$$

for $n=0, 1, 2, \cdots$.

If $\{\lim_n x_n\} \ni x$, we call x a *limit* of sequence $\{x_n\}$.

Then the following propositions hold:

(1.2.2) **Proposition.** *Let* $\{x_{n_i}\}$ *be an arbitrary subsequence of a sequence* $\{x_n\}$ *in a ranked space* E. *If* $\{\lim_n x_n\} \ni x$, *then*

$$\{\lim_i x_{n_i}\} \ni x.$$

(1.2.3) **Proposition.** *Let* $\{x_n\}$ *be a sequence in a ranked space* E. *If* $\{\lim_n x_{m+n}\} \ni x$, *where* m *is a positive integer, then*

$$\{\lim_n x_n\} \ni x.$$

(1.2.4) **Proposition.** *Let* $\{x_n\}$ *be a sequence in a ranked space* E *such that* $x_n = x$ *for* $n = 0, 1, 2, \cdots$. *Then*

$$\{\lim_n x_n\} \ni x.$$

In fact, let us check (1.2.4), the others being obvious. Since $\mathfrak{B}(x) = \{V(x)\} \neq \phi$, we can choose a neighborhood $U(x) \in \mathfrak{B}(x)$. By the assumption that E is a ranked space we can find an integer α_0 and a neighborhood $V_0(x) \in \mathfrak{B}(x)$ such that $V_0(x) \in \mathfrak{B}_{\alpha_0}$, $V_0(x) \subset U(x)$. Let $\beta_1 = \max(\alpha_0, 1)$, then we can find an integer α_1 and a neighborhood $V_1(x) \in \mathfrak{B}(x)$ such that $\alpha_1 \geq \beta_1$, $V_1(x) \in \mathfrak{B}_{\alpha_1}$ and $V_1(x) \subset V_0(x)$. Let $\beta_2 = \max(\alpha_1, 2)$, then analogously we can find an integer α_2 and a neighborhood $V_2(x) \in \mathfrak{B}(x)$ such that $\alpha_2 \geq \beta_2$, $V_2(x) \in \mathfrak{B}_{\alpha_2}$, and $V_2(x) \subset V_1(x), \cdots$.

Continuing this process, we obtain a sequence $\{V_n(x)\}$ of neighborhoods and a sequence $\{\alpha_n\}$ of integers such that

$$V_0(x) \supset V_1(x) \supset V_2(x) \supset \cdots \supset V_n(x) \supset \cdots, \quad 0 \leq n < \omega_0,$$
$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \cdots, \quad 0 \leq n < \omega_0,$$
$$\sup_n \alpha_n = \omega_0, \quad V_n(x) \ni x_n = x \text{ and } V_n(x) \in \mathfrak{B}_{\alpha_n},$$

for $n = 0, 1, 2, \cdots$.

$$\therefore \quad \{\lim x_n\} \ni x.$$

1.3. Continuity. Let E_1, E_2 be ranked spaces and $f: E_1 \to E_2$ a map from E_1 into E_2.

(1.3.1) **Definition.** We say that a map $f: E_1 \to E_2$ is *continuous* at the point a if and only if

$$\{\lim x_n\} \ni a \Rightarrow \{\lim f(x_n)\} \ni f(a).$$

$f: E_1 \to E_2$ continuous means that it is continuous at each point of E_1. One easily verifies that the compose of continuous maps is also continuous. $f: E_1 \to E_2$ is called a *homeomorphism* if and only if $f: E_1 \to E_2$ is bijective and $f: E_1 \to E_2$ as well as $f^{-1}: E_2 \to E_1$ are continuous.

1.4. Separated ranked space. When a sequence $\{x_n\}$ converges to x in a ranked space E, it is possible that $\{\lim x_n\} \ni x$, $\{\lim x_n\} \ni y$ and $x \neq y$. In order to get rid of these cases we introduce the following axiom [3].

(1.4.1) **Axiom** (T_0). *Let* E *be a ranked space with the indicator* ω_0. *Then, for any elements* $x, y \in E$ *with* $x \neq y$, *there exists an integer* $\alpha(x, y)$ $(0 \leq \alpha(x, y) < \omega_0)$ *such that for any integers* m, n *with* $m, n \geq \alpha(x, y)$ *and for any neighborhoods* $V(x) \in \mathfrak{B}(x)$, $V(y) \in \mathfrak{B}(y)$,

$$V(x) \in \mathfrak{B}_m, \ V(y) \in \mathfrak{B}_n \Rightarrow V(x) \cap V(y) = \phi.$$

(1.4.2) **Definition.** A ranked space which satisfies the axiom (T_0) is called a *separated ranked space*.

Then the following proposition is easily proved.

(1.4.3) **Proposition.** *Suppose that Axiom* (T_0) *holds in a ranked space* E *and let* $\{x_n\}$ *be a sequence in* E. *If* $\{\lim x_n\} \ni x$ *and* $\{\lim x_n\} \ni y$, *then*

$$x=y.$$

By this Proposition, if a ranked space E satisfies Axiom (T_0) and $\{\lim x_n\} \ni x$, then the limit of the sequence $\{x_n\}$ is uniquely determined. So in this case we may write

$$\lim_n x_n = x$$

instead of $\{\lim x_n\} \ni x$.

1.5. Direct product of ranked spaces. Let $E_1, E_2, \cdots, E_m$ be a family of ranked spaces with the indicator ω_0, i.e., with every element $x \in E_i$ there is associated a non-empty set $\mathfrak{B}_{E_i}(x)=\{V(x)\}$ satisfying Condition (1.1.1) and further in each space E_i there is a countable system $\mathfrak{B}_0(E_i), \mathfrak{B}_1(E_i), \mathfrak{B}_2(E_i), \cdots, \mathfrak{B}_n(E_i), \cdots$ of families of neighborhoods such that, for any $x \in E_i$, $V(x) \in \mathfrak{B}_{E_i}(x)$ and for an integer n $(0 \leq n < \omega_0)$, there exists an integer l and a neighborhood $U(x) \in \mathfrak{B}_{E_i}(x)$ satisfying the following conditions:

$$l \geq n, \quad U(x) \in \mathfrak{B}_l(E_i) \text{ and } U(x) \subset V(x).$$

We denote by $E_1 \times E_2 \times \cdots \times E_m$ (or $\times E_i$) the set of all elements $(x_1, x_2, \cdots, x_m)$, where $x_1 \in E_1$, $x_2 \in E_2$, $\cdots$, $x_m \in E_m$. If $E_1 = E_2 = \cdots = E_m = E$, we denote by E^m instead of $\times E_i$.

We now define a neighborhood system $\mathfrak{B}_{\times E_i}(z)$ to each point $z = (x_1, x_2, \cdots, x_m) \in \times E_i$ as follows:

$$\mathfrak{B}_{\times E_i}(z)=\{V_1(x_1)\times V_2(x_2)\times \cdots \times V_m(x_m);$$
$$V_1(x_1) \in \mathfrak{B}_{E_1}(x_1),\ V_2(x_2) \in \mathfrak{B}_{E_2}(x_2),\ \cdots,\ V_m(x_m) \in \mathfrak{B}_{E_m}(x_m)\}.$$

Then it is obvious that $\times E_i$ is a neighborhood space, i.e., it satisfies Condition (1.1.1).

We now define a countably system $\mathfrak{B}_0(\times E_i)$, $\mathfrak{B}_1(\times E_i)$, $\cdots$, $\mathfrak{B}_n(\times E_i)$, $\cdots$ in the following way:

$$\mathfrak{B}_n(\times E_i)=\{V_1 \times V_2 \times \cdots \times V_m;\ V_1 \in \mathfrak{B}_{\alpha_1}(E_1),\ V_2 \in \mathfrak{B}_{\alpha_2}(E_2),\ \cdots,$$
$$V_m \in \mathfrak{B}_{\alpha_m}(E_m) \text{ and } n=\min(\alpha_1, \alpha_2, \cdots, \alpha_m)\}$$

for $n=0, 1, 2, \cdots$.

Then $E_1 \times E_2 \times \cdots \times E_m$ is a ranked space with the indicator ω_0. In fact, for any $z=(x_1, x_2, \cdots, x_m) \in \times E_i$, $W(z)=V_1(x_1)\times V_2(x_2)\times \cdots \times V_m(x_m) \in \mathfrak{B}_{\times E_i}(z)$ and for an integer n $(0 \leq n < \omega_0)$, since E_i is a ranked space, there is an integer α_i and a neighborhood $U_i(x_i) \in \mathfrak{B}_{E_i}(x_i)$ such that

$$\alpha_i \geq n, \quad U_i(x_i) \in \mathfrak{B}_{\alpha_i}(E_i) \text{ and } U_i(x_i) \subset V_i(x_i).$$

for $i=1, 2, \cdots, m$.

Let

$$W'(z)=U_1(x_1)\times U_2(x_2)\times \cdots \times U_m(x_m)$$

and $p=\min(\alpha_1, \alpha_2, \cdots, \alpha_m)$, then we have

$$p \geq n,\ W'(z) \subset W(z),\ W'(z) \in \mathfrak{B}_{\times E_i}(z) \text{ and } W'(z) \in \mathfrak{B}_p(\times E_i).$$

Therefore $E_1 \times E_2 \times \cdots \times E_m$ is a ranked space. We shall call $E_1 \times E_2 \times \cdots \times E_m$ (or $\times E_i$) the direct product of ranked spaces $E_1, E_2, \cdots, E_m$. If $E_1 = E_2 = \cdots = E_m = E$, we denote by E^m instead of $\times E_i$.

In the direct product $\times E_i$ the following proposition holds:

(1.5.1) **Proposition.** *Let* $\{z_n\} = \{(x_{n1}, x_{n2}, \cdots, x_{nm})\}$ *be a sequence in a direct product* $E_1 \times E_2 \times \cdots \times E_m$ *of ranked spaces* $E_1, E_2, \cdots, E_m$ *and* $z = (x_1, x_2, \cdots, x_m) \in \times E_i$, *then* $\{\lim z_n\} \ni z$ *if and only if* $\{\lim x_{nk}\} \ni x_k$ *for* $k = 1, 2, \cdots, m$.

Proof. (a) Suppose that $\{\lim z_n\} \ni z$, i.e., there exists a sequence $\{W_n(z)\}$ of neighborhoods of z and a sequence $\{\gamma_n\}$ of integers such that

$$W_0(z) \supset W_1(z) \supset W_2(z) \supset \cdots \supset W_n(z) \supset \cdots,\quad 0 \leq n < \omega_0,$$
$$\gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n \leq \cdots,\quad 0 \leq n < \omega_0,$$
$$\sup_n \gamma_n = \omega_0,\quad W_n(z) \ni z_n, \text{ and } W_n(z) \in \mathfrak{B}_{\gamma_n}(\times E_i),$$

for $n = 0, 1, 2, \cdots$.

Let

$$W_n(z) = V_{n1}(x_1) \times V_{n2}(x_2) \times \cdots \times V_{nm}(x_m),$$

where $n = 0, 1, 2, \cdots$.

By assumption we have

$$V_{n1}(x_1) \in \mathfrak{B}_{E_1}(x_1),\ V_{n2}(x_2) \in \mathfrak{B}_{E_2}(x_2),\ \cdots,\ V_{nm}(x_m) \in \mathfrak{B}_{E_m}(x_m),$$
$$V_{n1}(x_1) \in \mathfrak{B}_{\alpha_{n1}}(E_1),\ V_{n2}(x_2) \in \mathfrak{B}_{\alpha_{n2}}(E_2),\ \cdots,\ V_{nm}(x_m) \in \mathfrak{B}_{\alpha_{nm}}(E_m),$$

and $$\gamma_n = \min(\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nm}).$$

Since $W_n(z) \supset W_{n+1}(z)$, we have

$$V_{nk}(x_k) \supset V_{(n+1)k}(x_k),$$

for $n = 0, 1, 2, \cdots$ and $k = 1, 2, \cdots, m$.

Further $W_n(z) \ni z_n$ implies $V_{nk}(x_k) \ni x_{nk}$.

Thus we have

$$V_{0k}(x_k) \supset V_{1k}(x_k) \supset V_{2k}(x_k) \supset \cdots \supset V_{nk}(x_k) \supset \cdots,\quad 0 \leq n < \omega_0,$$
$$V_{nk}(x_k) \ni x_{nk},\ V_{nk}(x_k) \in \mathfrak{B}_{\alpha_{nk}}(E_k) \text{ and } \alpha_{nk} \geq \gamma_n,$$

where $n = 0, 1, 2, \cdots$ and $k = 1, 2, \cdots, m$.

Since $\sup_n \gamma_n = \omega_0$, we can find a subsequence $\{\alpha_{n_i k}\}$ of $\{\alpha_{nk}\}$ such that

$$\alpha_{n_0 k} < \alpha_{n_1 k} < \alpha_{n_2 k} < \cdots < \alpha_{n_i k} < \cdots,\quad 0 \leq i < \omega_0.$$

Here we may assume that $n_0 = 0$.

We now define two sequences $\{U_{nk}(x_k)\}$ and $\{\beta_{nk}\}$ as follows:

$$\begin{array}{llll}
U_{0k}(x_k) = V_{0k}(x_k) & \ni x_{0k} & \mathfrak{B}_{\alpha_{0k}}(E_k) & \beta_{0k} = \alpha_{0k} \\
U_{1k}(x_k) = V_{0k}(x_k) & \ni x_{1k} & \mathfrak{B}_{\alpha_{0k}}(E_k) & \beta_{1k} = \alpha_{0k} \\
\cdots\cdots & & & \\
U_{(n_1-1)k}(x_k) = V_{0k}(x_k) & \ni x_{(n_1-1)k} & \mathfrak{B}_{\alpha_{0k}}(E_k) & \beta_{(n_1-1)k} = \alpha_{0k} \\
U_{n_1 k}(x_k) = V_{n_1 k}(x_k) & \ni x_{n_1 k} & \mathfrak{B}_{\alpha_{n_1 k}}(E_k) & \beta_{n_1 k} = \alpha_{n_1 k} \\
U_{(n_1+1)k}(x_k) = V_{n_1 k}(x_k) & \ni x_{(n_1+1)k} & \mathfrak{B}_{\alpha_{n_1 k}}(E_k) & \beta_{(n_1+1)k} = \alpha_{n_1 k} \\
\cdots\cdots & & &
\end{array}$$

$$U_{(n_2-1)k}(x_k)=V_{n_1k}(x_k) \quad \ni x_{(n_2-1)k} \quad \mathfrak{V}_{\alpha_{n_1k}}(E_k) \quad \beta_{(n_2-1)k}=\alpha_{n_1k}$$
$$U_{n_2k}(x_k)=V_{n_2k}(x_k) \quad \ni x_{n_2k} \quad \mathfrak{V}_{\alpha_{n_2k}}(E_k) \quad \beta_{n_2k}=\alpha_{n_2k}$$
$$\cdots\cdots\cdots$$

Then we have

$$U_{nk}(x_k) \in \mathfrak{V}_{E_k}(x_k),$$
$$U_{0k}(x_k) \supset U_{1k}(x_k) \supset U_{2k}(x_k) \supset \cdots \supset U_{nk}(x_k) \supset \cdots, \quad 0 \leq n < \omega_0,$$
$$\beta_{0k} \leq \beta_{1k} \leq \beta_{2k} \leq \cdots \leq \beta_{nk} \leq \cdots, \quad 0 \leq n < \omega_0,$$
$$\sup_n \beta_{nk}=\omega_0, \quad U_{nk}(x_k) \ni x_{nk}, \text{ and } U_{nk}(x_k) \in \mathfrak{V}_{\beta_{nk}}(E_k).$$
$$\therefore \quad \{\lim_n x_{nk}\} \ni x_k,$$

where $k=1, 2, \cdots, m$.

(b) Suppose conversely that $\{\lim_n x_{nk}\} \ni x_k$ for $k=1, 2, \cdots, m$ i.e., there exists a sequence $\{V_{nk}(x_k)\}$ of neighborhoods of x_k and a sequence $\{\alpha_{nk}\}$ of integers such that

$$V_{0k}(x_k) \supset V_{1k}(x_k) \supset V_{2k}(x_k) \supset \cdots \supset V_{nk}(x_k) \supset \cdots, \quad 0 \leq n < \omega_0,$$
$$\alpha_{0k} \leq \alpha_{1k} \leq \alpha_{2k} \leq \cdots \leq \alpha_{nk} \leq \cdots, \quad 0 \leq n < \omega_0,$$
$$\sup_n \alpha_{nk}=\omega_0, \quad V_{nk}(x_k) \ni x_{nk}, \text{ and } V_{nk}(x_k) \in \mathfrak{V}_{\alpha_{nk}}(E_k),$$

for $n=0, 1, 2, \cdots$ and $k=1, 2, \cdots, m$.

Put

$$W_0(z)=V_{01}(x_1) \times V_{02}(x_2) \times \cdots \times V_{0k}(x_k) \times \cdots \times V_{0m}(x_m)$$
$$W_1(z)=V_{11}(x_1) \times V_{12}(x_2) \times \cdots \times V_{1k}(x_k) \times \cdots \times V_{1m}(x_m)$$
$$\cdots\cdots\cdots$$
$$W_n(z)=V_{n1}(x_1) \times V_{n2}(x_2) \times \cdots \times V_{nk}(x_k) \times \cdots \times V_{nm}(x_m)$$
$$\cdots\cdots\cdots$$

then since $V_{ik}(x_k) \supset V_{(i+1)k}(x_k)$, where $k=1, 2, \cdots, m$ and $i=0, 1, 2, \cdots$. We have

$$W_0(z) \supset W_1(z) \supset W_2(z) \supset \cdots \supset W_n(z) \supset \cdots, \quad 0 \leq n < \omega_0.$$

Let

$$\gamma_p=\min(\alpha_{p1}, \alpha_{p2}, \cdots, \alpha_{pm})$$

for $p=0, 1, 2, \cdots$, then it follows, using $\alpha_{ik} \leq \alpha_{(i+1)k}$ ($k=1, 2, \cdots, m$ and $i=0, 1, 2, \cdots$), that

$$\gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n \leq \cdots, \quad 0 \leq n < \omega_0.$$

Being $\sup_n \alpha_{nk}=\omega_0$ ($k=1, 2, \cdots, m$), we get

$$\sup_n \gamma_n=\omega_0.$$

It is obvious that we have

$$W_n(z) \in \mathfrak{V}_{\gamma_n}(\times E_i) \text{ and } W_n(z) \ni z_n, \quad \text{for } n=0, 1, 2, \cdots.$$

Therefore

$$\{\lim z_n\} \ni z.$$

If $E_1, E_2, \cdots, E_m$ are a family of separated ranked spaces, then it is clear that $\times E_i$ is also a separated ranked space.

References

[1] K. Kunugi: Sur les espaces complets et régulièrement complets. I. Proc. Japan Acad., **30**, 553–556 (1954).
[2] ——: Sur la méthode des espaces rangés. I. Proc. Japan Acad., **42**, 318–322 (1967).
[3] ——: loc. cit.