47. On a Kind of Uniqueness of Set-Entourage Uniformities for Function Spaces

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Let $\mathfrak{W}_p$, $\mathfrak{W}_c$, and $\mathfrak{W}_u$ be respectively the uniformity of pointwise, compact, and uniform convergence on a family of mappings of a uniform space into another uniform space. Usually uniformities $\mathfrak{W}_p$ and $\mathfrak{W}_u$ are simpler than $\mathfrak{W}_c$ for our use, but defective in some respect —$\mathfrak{W}_p$ is in many ways unnatural (cf. [3], p. 219) and $\mathfrak{W}_u$ can be applied only to families of uniformly continuous mappings when the continuity of operation of mapping composition is required (cf. Theorems 4 and 5 of [2]). On the other hand, as for $\mathfrak{W}_c$ there are no such defects as above and many desirable properties have been found in the literature. It seems to us that $\mathfrak{W}_c$ is the most natural set-entourage uniformity for families of mappings.

Let X be, for example, any locally euclidean, uniformly locally connected, metric space or any convex subset of a normed space, and let $\mathfrak{C}$ be the family of all continuous mappings of X into itself. The purpose of this note is to show that any set-entourage uniformity on $\mathfrak{C}$ must coincides with $\mathfrak{W}_c$, if the joint continuity and the continuity of operation of mapping composition are required under the uniform topology. This is an answer to the problem proposed in [2].

We begin with the definition of two terms used in the main theorem.

Definition 1. A uniform space X endowed with a uniformity $\mathfrak{U}$ is *uniformly deformable* if for any entourage $U \in \mathfrak{U}$ there exists an entourage $U^* \in \mathfrak{U}$ as follows: for any two U^*-close points p and q, there exists a continuous mapping f of X into itself such that $f(p)=q$ and $(x, f(x)) \in U$ for any $x \in X$.

Examples. The following spaces i), ii), $\cdots$, v) are uniformly deformable uniform spaces, whereas the space vi) is a manifold that is not uniformly deformable: i) locally euclidean, uniformly locally connected, uniform spaces, ii) locally euclidean, compact, uniform spaces, iii) convex subsets of a normed space, iv) the set of all rational points in a euclidean space, v) discrete uniform spaces, and vi) the set of all points (x, y) in the euclidean plane such that $(x^2+1)y^2>x$. (The uniformities of iii), iv), and vi) are the usual ones.)

Definition 2. A topological space is *locally dense* if there exists a family of dense-in-itself open sets which forms an open base for the topology of the space.

Now we prove the main theorem.

Theorem 1. *Let X be a uniformly deformable, locally dense, locally compact, metric space that contains a non-degenerate arc, $\mathfrak{F}$* (resp. $\mathfrak{C}$) *the family of all mappings* (resp. *all continuous mappings*) *of X into itself, $\mathfrak{W}_c$ the uniformity of compact convergence on $\mathfrak{F}$, and $\mathfrak{W}$ any set-entourage uniformity on $\mathfrak{F}$. Then $\mathfrak{W}$ coincides with $\mathfrak{W}_c$ on $\mathfrak{C}$ if and only if the mapping $(u, x) \to u(x)$ of $\mathfrak{C} \times X$ into X and the mapping $(u, v) \to u \circ v$ of $\mathfrak{C} \times \mathfrak{C}$ into $\mathfrak{C}$ are continuous with respect to the relative topology on $\mathfrak{C}$ induced by the uniformity $\mathfrak{W}$.*

Proof. The necessity under the condition that X is locally compact is well-known (cf. [1]). We prove the sufficiency. Let $\mathfrak{S}$ be the family of subsets of X which defines the uniformity $\mathfrak{W}$, and K be a non-degenerate arc in X. Let A be any set belonging to $\mathfrak{S}$. Every continuous mapping f of $\bar{A}$ into K has a continuous extension to X by the extension theorem of Tietze. Since the operation of composition of mappings belonging to $\mathfrak{C}$ is continuous relatively, it is seen by § 5 of [2] that

i) f must be uniformly continuous on $\bar{A}$, and

ii) there exist an entourage U of the metric uniformity on X and a set $B \in \mathfrak{S}$ such that $U(\bar{A}) \subset \bar{B}$.

The fact that i) has been shown implies that for each set $C \in \mathfrak{S}$ the derived set of $\bar{C}$ is compact (cf. [4]). Since X is locally dense, there exists a dense-in-itself neighborhood V of A such that $V \subset U(\bar{A})$, and so $\bar{A} \subset V \subset \bar{B}'$ for B in ii), where $\bar{B}'$ is the derived set of $\bar{B}$. Hence $\bar{A}$ is compact. Consequently we have shown that $\mathfrak{W}_c$ is finer than the uniformity $\bar{\mathfrak{W}}$ of $\bar{\mathfrak{S}}$-convergence, where $\bar{\mathfrak{S}} = \{\bar{A} \mid A \in \mathfrak{S}\}$. Now by the joint continuity on $\mathfrak{C} \times X$, $\bar{\mathfrak{W}}$ is finer than $\mathfrak{W}_c$ (cf. Theorems 2 and 3 of [2]). Finally the fact that $\bar{\mathfrak{W}}$ coincides with $\mathfrak{W}$ on $\mathfrak{C} \times \mathfrak{C}$ ([1], p. 280) completes the proof.

Corollary. *If X is a locally euclidean, uniformly locally connected, metric space or a convex subset of a normed space, then the conclusion of Theorem* 1 *is valid.*

Remark. If X is a *compact* uniform space that contains a non-degenerate arc, then $\mathfrak{W}$ coincides with $\mathfrak{W}_c$ on $\mathfrak{C}$ if and only if $\mathfrak{W}$ gives the joint continuity on $\mathfrak{C} \times X$. In fact $\bar{A}$ is compact for any $A \in \mathfrak{S}$, and so $\mathfrak{W}_c$ is finer than $\bar{\mathfrak{W}}$. The remaining parts of the proof are the same as those in Theorem 1.

Theorem 2. *Let X be a locally dense, locally compact, metric space that contains a non-degenerate arc, and let $\mathfrak{F}, \mathfrak{C}, \mathfrak{W}_c$, and $\mathfrak{W}$ be* *the same as those in Theorem* 1. *Then* $\mathfrak{W}$ *coincides with* $\mathfrak{W}_c$ *if and only if the mapping* $(u, x)\to u(x)$ *of* $\mathfrak{F}\times X$ *into* X *and the mapping* $(u, v)\to u\circ v$ *of* $\mathfrak{F}\times\mathfrak{F}$ *into* $\mathfrak{F}$ *are continuous on* $\mathfrak{C}\times X$ *and* $\mathfrak{C}\times\mathfrak{C}$ *respectively with respect to the topology on* $\mathfrak{F}$ *induced by* $\mathfrak{W}$.

Proof. A slight modification of the proof in Theorem 1, using Theorems 2 and 4 in [2], gives the proof.

References

[1] N. Bourbaki: General Topology. Part 2, translated from French, Hermann and Addison-Wesley (1966).
[2] T. Karube: Uniformities for function spaces and continuity conditions (to appear).
[3] J. L. Kelly: General Topology. D. Van Nostrand (1955).
[4] N. Levine and W. G. Saunders: Uniformly continuous sets in metric spaces. Amer. Math. Monthly, **67**, 153–156 (1960).