46. *Extended p-th Powers of Complexes and Applications to Homotopy Theory*

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1. Extended p-th power of a complex. Throughout this note p will denote an odd prime, $m=(p-1)/2$, $\pi=Z_p$ a cyclic group of order p, and the homology and cohomology groups will have the coefficient group Z_p. Let $W=W^\infty(=S^\infty)$ be a regular π-free acyclic CW-complex having one π-free basic cell e_i for each dimension i. The cells e_i are oriented such that in the infinite dimensional lens space W/π the dual $w_i \in H^i(W/\pi)$ of the class of e_i satisfies $w_{2i}=(w_i)^2$ and $\beta(w_2)=w_1$ for the cohomology Bockstein β.

For a finite CW-complex X, the product and the reduced join of p-copies of X will be denoted by $X^p=X\times\cdots\times X$ and $X^{(p)}=X\wedge\cdots\wedge X$ respectively. Let π acts on X^p and $X^{(p)}$ as cyclic permutations of the factors, and consider the quotient complexes

$$W^r\times_\pi X^p \quad\text{and}\quad ep^r(X)=(W^r\times_\pi X^{(p)})/(W^r/\pi),$$

where W^r indicates the r-skeleton of X and $W^r/\pi=W^r\times_\pi x_0^{(p)}$ for the base point x_0 of X. Let $x_0, x_1, x_2, \cdots$ be a Z_p-basis of homogeneous elements of $H_*(X)$ which satisfies that if $\varDelta x_j\neq 0$ for the homology Bockstein then $\varDelta x_j=x_l$ for some l. A Z_p-basis of $H_*(W\times_\pi X^p)$ is given as the classes represented by the following cycles (cf. [2], [3]):

$$e_i\otimes_\pi x_j^p,\quad j=0,1,2,\cdots,\quad x_j^p=x_j\otimes\cdots\otimes x_j\ (p\text{-times}),$$
$$e_0\otimes_\pi(x_{j_1}\otimes\cdots\otimes x_{j_p}),\quad j_s\neq j_t\ \text{ for some } s,t,$$

where $(j_1,\cdots,j_p)$ runs through each representatives of the classes obtained by cyclic permutations of the indices. The same result holds for $H_*(W^r\times_\pi X^p)$ restricting e_i by $0\leq i\leq r$ and by adding cycles of the form $\partial(e_{r+1}\otimes_\pi(x_{j_1}\otimes\cdots\otimes x_{j_p}))$.

By the natural projection $W^r\times_\pi X^p\to ep^r(X)$, a Z_p-basis of $\tilde H_*(ep^r(X))$ is obtained from that of $H_*(W^r\times_\pi X^p)$ by omitting the cycles containing x_0.

Denote by $P^i_*: H_q\to H_{q-2i(p-1)}$ the dual of the Steenrod reduced power P^i, and let $P^i_*x_k=\Sigma_j a_{k,j}(i)x_j$ for $a_{k,j}\in Z_p$. Then the following relation has been established in [3].

Theorem 1. (*Nishida*).

$$P^n_*(e_{c+2n(p-1)}\otimes_\pi x_k^p)=\Sigma_{i,j}\binom{[c/2]+qm}{n-pi}a_{k,j}(i)(e_{c+2ip(p-1)}\otimes_\pi x_j^p)$$

$$-\mu(q)\varepsilon(c+1)\Sigma_{i,j}\binom{[(c+1)/2]+qm-1}{n-pi-1}a_{l,j}(i)(e_{c+p+2ip(p-1)}\otimes_\pi x_j^p),$$

where c may be negative, $q=\deg x_k$, $m=(p-1)/2$, $\mu(t)=(-1)^{mt}/m!$, $\varepsilon(s)=1$ if s is even, $\varepsilon(s)=0$ if s is odd, $x_l=\Delta x_k$ if $\Delta x_k\neq 0$, and the second term is omitted if $\Delta x_k=0$.

As is easily seen, $\Delta(e_s\otimes_\pi x_j^p)=\varepsilon(s)e_{s-1}\otimes_\pi x_j^p$.

For a base point preserving cellular map $f: X\to Y$, the product $1\times f^p: W^r\times X^p\to W^r\times Y^p$ defines a cellular map

$$ep^r(f): ep^r(X)\to ep^r(Y).$$

Obviously, *$ep^r(f)|ep^s(X)=ep^s(f)$, $s\leqq r$, $ep^0(f)=f\wedge\cdots\wedge f$ (p-times), $ep^r(1)=1$, $ep^r(g)\circ ep^r(f)=ep^r(g\circ f)$, and if $f\simeq f'$ (homotopic) then $ep^r(f)\simeq ep^r(f')$.*

Denote by $C_f=Y\cup_f CX$ the mapping cone of f and represents each point of CX by (x,t), $x\in X$, $t\in I=[0,1]$, with $(x,1)=(x_0,t)=y_0$, and $(x,0)=f(x)$. Then the natural inclusion of $ep^r(Y)$ into $ep^r(C_f)$ can be extended over a map (not cellular)

$$D_f: C_{ep^r(f)}=ep^r(Y)\cup C(ep^r(X))\to ep^r(C_f)$$

by sending $(w\times(x_1\wedge\cdots\wedge x_p),t)$ to $w\times((x_1,t)\wedge\cdots\wedge(x_p,t))$. Assume that the induced chain map $f_\#: C_*(X)\otimes Z_p\to C_*(Y)\otimes Z_p$ is trivial, hence so is $ep^r(f)_\#$. Then there is a canonical splitting $H_{q+1}(C_f)=H_q(X)+H_{q+1}(Y)$. Denote by $\hat{x}=x^\wedge\in H_{q+1}(C_f)$ the element corresponding to $x\in H_q(X)$, and similarly for $C_{ep^r(f)}$. Then we have

Theorem 2. *$D_{f*}((e_i\otimes_\pi x^p)^\wedge)=-\mu(q+1)(e_{i-p+1}\otimes_\pi(\hat{x})^p)$ $(=0$ if $i<p-1)$.*

In order to prove this, consider the diagonal map $d: W^r\times I\to W^r\times I^p$. Leaving fix d on $W^r\times\partial I$, we can deform d equivariantly toa cellular map d'. Put $D'_f(w\times(x_1\wedge\cdots\wedge x_p),t)=w'\times((x_1,t_1)\wedge\cdots\wedge(x_p,t_p))$ for $d'(w,t)=(w',t_1,\cdots,t_p)$. Then $D_f\simeq D'_f$. Let $d'_\#(e_i\otimes_\pi I)=\lambda\cdot e_{i-p+1}\otimes_\pi I^p+\cdots$, where each of the rest terms contains a face of I^p. Then, by use of the assumption $f_\#=0$, we have $D_{f*}((e_i\otimes_\pi x^p)^\wedge)=D'_{f*}((e_i\otimes_\pi x^p)^\wedge)=\pm\lambda\cdot e_{i-p+1}\otimes_\pi(\hat{x})^p$. Here the sign $\pm=(-1)^{qp(p-1)/2}=(-1)^{mq}$ is caused of the permutation $(X\times I)^p\to X^p\times I^p$ and the coefficient λ is $(-1)^m m!$ by Lemma 5.3 of [4, VII] by considering the case $C_f=S^1$. This proves Theorem 2.

2. Special cases. In the following, n will be sufficiently large so that complexes and maps considered are in stable range. S^n denotes an n-sphere, $M_p^{n+1}=s^n\cup_p e^{n+1}$ a Moore space of type (Z_p,n), and $i: S^n\to M_p^{n+1}$ and $\pi: M_p^n\to S^n$ the natural maps. We shall write sometimes the stable homotopy class of a map $f: X\to Y$ by the same symbol $f\in\{X,Y\}=\lim[S^nX,S^nY]$. For example, $\delta=i\pi\in\{M_p^n,M_p^{n+1}\}$, and a generator α of $\{M_p^{n+2p-2},M_p^n\}\approx Z_p$ is characterized by the relation $P_*^1e^{n+2p-1}=e^{n+1}$ in the mapping cone C_α of α. $G_t=\{S^{n+t},S^n\}$ is the

t-stem group, and $\alpha_1 = \pi\alpha i \in G_{2p-3}$ is the first element of order p.

First consider the complex $ep^{2r}(S^n)$ which consists of a vertex x_0 and cells e^{pn+j}, $0 \leq j \leq 2r$, with $\partial(e^{pn+2i}) = p \cdot e^{pn+2i-1}$. Up to homotopy type, $ep^{2r}(S^n)$ is a mapping cone of a map $M_p^{pn+2r-1} \to ep^{2r-2}(S^n)$. Using the results on the stable groups, we have

Lemma 1. *$ep^{4p-6}(S^n)$ has the same homotopy type as the bouquet of some mapping cones $S^{pn} \cup CM_p^{pn+2p-3}$ and $M_p^{pn+2i} \cup CM_p^{pn+2i+2p-3}$, $1 \leq i \leq p-2$. In particular, S^{pn} is a retract of $ep^{p-1}(S^n)$ and there exists a map of M_p^{pn+p-1} into $ep^{p-1}(S^n)$ inducing a monomorphism of the homology.*

Here the attaching maps of the above mapping cones are determined by P_*^1. By Theorem 1 we have $P_*^1(e^{pn+j+2p-2}) = ([j/2] + n(p-1)/2)e^{pn+j}$. In particular the attaching map of the first mapping cone is a multiple of $\pi\alpha\delta$ and it is trivial if and only if $n \equiv 0 \pmod p$. Thus we have

Lemma 2. *There exists a map of $ep^{4p-6}(S^n)$ into $C_{\alpha_1} = C_{i\alpha\pi} = S^{pn} \cup e^{pn+2p-2}$ which is identical on $ep^0(S^n) = S^{pn}$. If $n \equiv 0 \pmod p$ then we can replace C_{α_1} by S^{pn} and there exists a map of $M_p^{pn+2p-2}$ into $ep^{2p-2}(S^n)$ inducing a monomorphism of the homology.*

Next consider $ep^r(M_p^{n+1})$. For $x \in H_{n+1}(M_p^{n+1})$, we have by Theorem 1 $P_*^1(e_{p-2} \otimes_\pi x^p) = -\mu(n+2)e_0 \otimes_\pi (\Delta x)^p$ and $P_*^1(e_{p-1} \otimes_\pi x^p) = 0$. Thus we have

Lemma 3. *There exists a map of $C_{\pi\alpha} = S^{pn} \cup_{\pi\alpha} CM_p^{pn+2p-2}$ into $ep^{p-1}(M_p^{n+1})$ which is identical on $S^{pn} = ep^0(S^n) \subset ep^0(M_p^{n+1})$.*

Consider $\alpha_1 : S^{n+2p-3} \to S^n$ and the induced map $ep^r(S^{n+2p-3}) \to ep^r(S^n)$ for $r < 2p(p-1)$. In $ep^r(C_{\alpha_1})$ we see by Theorem 1 that $P_*^p(e_s \otimes_\pi x^p) = -\mu(n+2)\varepsilon(s+1)(e_{s-p} \otimes_\pi (P_*^1 x))$, $x \in H_{n+2p-2}(C_{\alpha_1})$. By Theorem 2, this gives a non-triviality of the functional P^p-operation for $ep^r(\alpha_1)$. In particular, we have

Lemma 4. *Let $\bar{\beta} : M_p^{pn+2p(p-1)-1} \to ep^{p-1}(S^{n+2p-3}) \to ep^{p-1}(S^n) \to S^{pn}$ be the composition of the map of Lemma 1 to $ep^{p-1}(S^{n+2p-3})$, $ep^{p-1}(\alpha_1)$ and the retraction of Lemma 1. Then $\bar{\beta} | S^{pn+2p(p-1)-2} = \beta_1$ is a generator of the p-component of $G_{2p(p-1)-2}$.*

Finally consider $\alpha i : S^{n+2p-2} \to M_p^{n+1}$ and $ep^{2p-2}(\alpha i)$ for the case $n \equiv 2 \pmod p$. Let $j : M_p^a \to ep^{2p-2}(S^{n+2p-2})$ be the map of Lemma 2, $a = pn + 2(p+1)(p-1)$. Denote by β_s a generator of the p-component of $G_{2(sp+s-1)(p-1)-2}$, $1 \leq s \leq p-1$.

Lemma 5. *For an element $\tilde{\beta}_{s-1}$ of $\pi_b(M_p^a)$, $b = pn + 2(sp+s-1) \times (p-1) - 2$, such that $\pi\tilde{\beta}_{s-1}\beta_s = {}_{-1}$, we have $ep^{2p-2}(\alpha i)_* j_* \beta_{s-1} \equiv i_* \beta_s$, $i : S^{pn} \subset ep^{2p-2}(S^n)$, modulo the images of $\pi_b(M_p^{pn+2})$ and $\pi_b(M_p^{pn+3})$, $2 \leq s \leq p-1$.*

The proof is based on the methods in [8], [9], but the details are too long to describe here.

3. Relations in stable homotopy. As before $\alpha_1 \in G_{2p-3}$ and $\beta_s \in G_{2(sp+s-1)(p-1)-2}$, $1 \leq s \leq p-1$, are elements of order p.

Theorem 3. *If $p \cdot \gamma = 0$ for $\gamma \in G_t$, then $\alpha_1 \gamma^p = 0$ and $\{\gamma^p, \alpha_1, p\iota\} \equiv 0$.*

Proof. By the assumption there exists a map $f: M_p^{n+1} \to S^{n-t}$ such that $f \mid S^n$ represents γ. Consider the composition of the map $C_{\pi\alpha} \to ep^{p-1}(M_p^{n+1})$ of Lemma 3, the induced map $ep^{p-1}(f): ep^{p-1}(M_p^{n+1}) \to ep^{p-1}(S^{n-t})$ and the retraction $ep^{p-1}(S^{n-t}) \to S^{pn-pt}$ of Lemma 1. Its restriction on S^{pn} represents γ^p. The existence of such a map is equivalent to $\gamma^p \pi\alpha = 0$ which indicates the last assertion, and $\alpha_1 \gamma^p = \gamma^p \alpha_1 = \gamma^p \pi \alpha i = 0$.

Theorem 4. *If $\alpha_1 \gamma = 0$ for $\gamma \in G_t$, then $\beta_1 \gamma^p = 0$ and $\{\gamma^p, \beta_1, p\iota\} \equiv 0$.*

Proof. By the assumption, $ep^{p-1}(\gamma) \circ ep^{p-1}(\alpha_1)$ is homotopic to zero. Let $j: M_p^{pn+2p(p-1)-1} \to ep^{p-1}(S^{n+2p-3})$ be the map of Lemma 1. Since $\{M_p^{pn+2p(p-1)-1}, M_p^{pn+2i}\} = 0$ for $1 \leq i < p-1$, Lemma 1 and Lemma 4 show that $ep^{p-1}(\alpha_1) \circ j$ is homotopic to $\bar{\beta}$. Then applying the retraction $ep^{p-1}(S^{n-t}) \to S^{pn-pt}$ of Lemma 1 we have that $\gamma^p \circ \bar{\beta}$ is homotopic to zero, and the theorem follows.

Theorem 5. *If $\{\alpha_1, p\iota, \gamma\} \equiv 0$ for $\gamma \in G_t$ and $2 \leq s \leq p-1$, then $\beta_s \gamma^p \equiv 0 \bmod \alpha_1\ G_c$, $c = pt + 2(sp+s-2)(p-1) - 1$.*

Proof. Remark that in Lemma 5 the generators of $\pi_b(M_p^{pn+2})$ and $\pi_b(M_p^{pn+3})$ are of the form $\xi\alpha_1$. By the assumption $\bar{\gamma} \circ \alpha i \simeq 0$ for an extension $\bar{\gamma}: M_p^{n+1} \to S^{n-t}$ of γ. Let $r: ep^{2p-2}(S^{n-t}) \to C_{\alpha_1} = S^{pn-pt} \cup e^{pn-pt+2p-2}$ be the map of Lemma 2. Then $r_* ep^{2p-2}(\bar{\gamma})_* ep^{2p-2}(\alpha i)_* j_* \beta_{s-1}$ and $r_* ep^{2p-2}(\pi_b(M_p^{pn+i}))$, $i = 2, 3$, vanish. Thus Lemma 5 shows $i_*(\gamma^p \beta_s) = r_* ep^{2p-2}(\bar{\gamma})_* i_* \beta_s = 0$, *i.e.*, $\beta_s \gamma^p = \gamma^p \beta_s \equiv 0 \bmod \alpha_1 \cdot G_c$.

For the case $\gamma = \beta_1$ and $s = 2$, we know that the p-component of G_c vanishes [7]. Thus

Corollary 1. *$\beta_2 \beta_1^p = 0$, and the p-component of the $(2(p^2+2p) \times (p-1) - 4)$-stem group vanishes.*

By Theorems 3 and 4, we have $\beta_1^{p^2+1} = 0$, but this is not best possible since $\beta_1^6 = 0$ for $p = 3$. If $p \geq 5$ and $2 \leq s \leq p-1$, then $\{\alpha_1, p\iota, \beta_s\} \equiv 0$. It follows from Theorems 5 and 3 that $\beta_s^{2p+1} = 0$. If $p = 3$ we have $\{\alpha_1, 3\iota, \beta_2\} = \pm \beta_1^3$, hence $\{\alpha_1, 3\iota, \beta_2^2\} \equiv \pm \beta_2 \beta_1^3 = 0$. Thus we have $\beta_2^{10} = 0$.

Corollary 2. *The elements β_s, $1 \leq s \leq p-1$, are nilpotent.*

Here we make some remarks. As in Lemma 4, for a map $p: S^n \to S^n$ of degree p, the map $ep^{2p-2}(p)$ composed with maps of Lemma 2 gives $\pi\alpha$. The composition of the map of Lemma 3 and $ep^{p-1}(\bar{\beta})$ for the map $\bar{\beta}$ of Lemma 4 has a non-trivial functional P^{p^2}-operation. This proves the main theorem of [7]. Further discussions give a complex $S^n \cup e^{n+a} \cup e^{n+b} \cup e^{n+b+1} \cup e^{n+c} \cup e^{n+c+1}$, $a = p^2(2p(p-1)$

$-2)+1$, $b=a+2p(p-1)-1$, $c=b+2p-1=2p^3(p-1)-1$, with $P^{p^3}H^n=\beta P^1P^pH^{n+a}=H^{n+c+1}$ and $\beta_1^{p^2}$ as the attaching map of e^{n+a}.

4. Non-associativity in mod 3 generalized cohomology theory. Let h^* be a generalized (reduced cohomology theory equipped with a commutative and associative multiplication μ. For an integer $q>1$, mod q cohomology theory $h^*(\ \ ;Z_q)$ is defined by $h^i(X;Z_q)=h^{i+2}\times(X\wedge M_q)$, M_q being a Moore space of type $(Z_q, 1)$. If $q\not\equiv 2 \pmod 4$, $M_q\wedge M_q$ is stably homotopy equivalent to $SM_q\vee S^2M_q$. Let $\psi: S^2M_q\to M_q\wedge M_q$ and $\varphi: M_q\wedge M_q\to SM_q$ be the natural maps composed the above equivalence. Then $\mu_q=\sigma^{-2}(1\wedge\psi)^*\mu$ gives an admissible multiplication in $h^*(\ \ ;Z_q)$ in the sense of [1].

We consider the case that q is odd, then $\psi\simeq T\psi$ for the switching permutation T of $M_q\wedge M_q$. This implies the commutativity of μ_q. Also, the associativity is deduced from $(1\wedge\psi)S^2\psi\simeq(\psi\wedge 1)(1\wedge T)S^2\psi$. By $\psi\simeq T\psi$ we have $(\psi\wedge 1)(1\wedge T)S^2\psi=P(1\wedge\psi)(S^2T)(S^2\psi)\simeq P(1\wedge\psi)S^2\psi$ for a cyclic permutation P of $M_q^{(3)}=M_q\wedge M_q\wedge M_q$. The stable group $\{S^4M_q, M_q^{(3)}\}$ is generated by the element $(1\wedge\psi)S^2\psi$ of order q and the composition $i\alpha_1\pi: S^4M_q\to S^6\to S^3\to M_q^{(3)}$ of order (q, 3). By the arguments of [1] we have $(1\wedge\psi)S^2\psi-P(1\wedge\psi)S^2\psi=k\cdot i\alpha_1\pi$ for some $k\in Z_{(q,3)}$ and this implies the relation $x(yz)-(xy)z=k\cdot\alpha_1^{**}(\delta_q x)(\delta_q y)(\delta_q z)$ in $h^*(\ \ ;Z_q]$.

Similarly, $S\varphi(1\wedge\varphi)-S\varphi(1\wedge\varphi)P=k'\cdot i'\alpha_1\pi'$ for some $k'\in Z_{(q,3)}$ and $i'\alpha_1\pi': M_q^{(3)}\to S^6\to S^3\to S^2M_q$, and this implies $x(yz)-(xy)z=k'\cdot\alpha_1(\partial_q x)\times(\partial_q y)(\partial_q z)$ for the multiplication in the stable group $\pi_*(M_q)$ given by φ. By use of $\varphi\psi=0$, $\pi=\pi'(1\wedge\psi)S^2\psi$, $i'=S\varphi(1\wedge\varphi)i$, we have $k\cdot i'\alpha_1\pi=-S\varphi(1\wedge\varphi)P(1\wedge\psi)S^2\psi=k'\cdot i'\alpha_1\pi$, and this implies $k\equiv k'$ (mod $(q, 3)$). Obviously $h^*(\ \ ;Z_q)$ and $\pi_*(M_q)$ are associative if $q\not\equiv 0 \pmod 3$ or if $q\equiv k\equiv 0 \pmod 3$.

Now assume $q\equiv 0 \pmod 3$, then $k\cdot i\alpha_1$ is an obstruction to extend the map $S\varphi(1\wedge\varphi)$ over $W^1\times_\pi M_q^{(3)}\supset M_q^{(3)}$ since $W^1\times_\pi M_q^{(3)}$ is obtained from $I\times M_q^{(3)}$ identifying $0\times M_q^{(3)}$ with $1\times M_q^{(3)}$ by the permutation P. It follows without difficulty that $k\not\equiv 0 \pmod 3$ if and only if $P^1_*(e_1\otimes_\pi x^3)\neq 0$ for a generator x of $H_2(M_q)$. By Theorem 1, $P^1_*(e_1\otimes_\pi x^3)=e_0\otimes_\pi(\Delta x)^3$. Thus $k\not\equiv 0 \pmod 3$ if and only if $\Delta x\neq 0$, *i.e.*, $q\not\equiv 0 \pmod 9$. Consequently we have

Theorem 6. *Let q be odd>1. If $q\not\equiv 0$ (mod 3) or $q\equiv 0$ (mod 9) then $h^*(\ \ ;Z_q)$ and $\pi_*(M_q)$ are associative. If $q\equiv 0$ (mod 3) and $q\not\equiv 0$ (mod 9), then we have $x(yz)-(xy)z=\pm\alpha_1^{**}(\delta_q x)(\delta_q y)(\delta_q z)$ in $h^*(\ \ ;Z_q)$ and $=\pm\alpha_1(\partial_q x)(\partial_q y)(\partial_q z)$ in $\pi_*(M_q.)$*

Note that $\pi_*(M_3)$ is not associative since $\alpha_1\beta_1^2\beta_2\neq 0$.

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