80. On Nuclear Spaces with Fundamental System of Bounded Sets. I

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We consider a nuclear space with a fundamental system of bounded sets. In this paper, we consider the open mapping and closed graph theorems in the nuclear space.

For nuclear spaces and its related notion see [8]. Most of the definitions and notations of locally convex spaces are taken from N. Bourbaki [1] and T. Husain [4].

1. In this section, we consider under what conditions nuclear space is the space with the open mapping and closed graph theorems.

Definition 1. *Let E be a locally convex vector space and E' its dual.*

(1) *ew^*-topology is defined to be the finest topology on E' which coincides with $\sigma(E', E)$ on each equicontinuous set of E'.*

(2) *p-topology is the $\mathfrak{S}$-topology on E' where $\mathfrak{S}$ consists of all precompact subsets of E.*

(3) *E is called a S-space if on E', $ew^*=p$.*

(4) *E is called a B-complete if a linear continuous and almost open mapping of E onto any locally convex vector space F is open.*

(5) *E is called a $B(\mathfrak{T})$-space if it satisfies the following statement; For each barrelled space F, a linear and continuous mapping of E onto F is open.*

Let E and F be normed spaces, U and V their closed unit balls respectively. A continuous linear mapping T of E in F is called a nuclear mapping if there exists a continuous linear form $a_n \in E'$ and $y_n \in F$ such that the following holds;

$$Tx = \sum_N <x, a_n> y_n \quad \text{for} \quad x \in E$$

and

$$\sum_N P_{U^0}(a_n) P_V(y_n) < +\infty.$$

For each nuclear mapping T define the norm:

$$\nu(T) = \inf\{\sum_N P_{U^0}(a_n) P_V(y_n)\}.$$

Let $\mathfrak{N}(E, F)$ be the set of all nuclear mappings of E into F, we introduce a norm in $\mathfrak{N}(E, F)$ by $\nu(T)$. Let $\mathcal{L}(E, F)$ be the set of all continuous linear mappings of E into F, let $\mathcal{A}(E, F)$ be the set of all mappings $t \in \mathcal{L}(E, F)$ such that $t(E)$ is a finite dimentional subspace of F, then we have the following

Proposition 1. *$\mathcal{A}(E, F)$ is a dense subspace in $\mathcal{N}(E, F)$.*

Proof. For any $T \in \mathcal{A}(E, F)$, it has the form

$$Tx = \sum_{n=1}^{K} <x, a_n> y_n, \quad a_1, \cdots, a_k \in E, \quad y_1, \cdots, y_k \in F.$$

Therefore $\mathcal{A}(E, \mathrm{F})$ is a linear subspace of $\mathcal{N}(E, F)$. Next, for any $T \in \mathcal{N}(E, F)$ there exists a continuous linear form $a_n \in E$ and elements $y_n \in F$ such that

$$Tx = \sum_N <x, a_n> y_n \quad \text{for} \quad x \in E,$$

and

$$\sum_N P_{U^0}(a_n) P_V(y_n) < +\infty.$$

Let $\mathcal{T}(N)$ be the family of all finite subsets of N, then for arbitrary $\delta > 0$ there exists $N_0 \in \mathcal{T}(N)$ such that

$$\sum_{N \setminus N_0} P_{U^0}(a_n) P_V(y_n) \leqq \delta.$$

Now we define the mapping $T_N \in \mathcal{A}(E, F)$ by

$$T_N x = \sum_N <x, a_n> y_n.$$

Then there exists N_0 such that

$$(T - T_N) \leqq \delta \quad \text{for all} \quad N \geqq N_0 (\in \mathcal{T}(N)).$$

This implies

$$\nu - \lim_N T_N = T.$$

The proof is complete.

Thus we have immediately the following

Corollary. *Each nuclear mapping is precompact.*

Let E be a locally convex space and U any closed and absolutely convex neighborhood of the origin in E. Let

$$P_U(x) = \inf\{\rho > 0 ; x \in \rho U\} \quad \text{for} \quad x \in E,$$

and

$$E(U) = E / \{x \in E ; P_U(x) = 0\}.$$

Then we introduce a topology on $E(U)$ by the norm

$$\| x(U) \| = P_U(x) \quad \text{for} \quad x(U) \in E(U)$$

where $x(U)$ corresponds to $x \in E$ in a natural way. For each closed and absolutely convex bounded subset A in locally convex space E, we define a linear subspace $E(A)$ of E by

$$E(A) = \{x \in E ; x \in fA \quad \text{for any} \quad \rho > 0\},$$

and topology of $E(A)$ is introduced by the norm

$$P_A(x) = \inf\{\rho > 0 ; x \in \rho A\} \quad \text{for} \quad x \in E(A).$$

The following proposition is an important assertion.

Proposition 2. *Any bounded subset of nuclear space E is precompact.*

Proof. We can assume without loss of generality that each bounded set is closed and absolutely convex bounded set, because there exists a closed and absolutely convex bounded set which contains a bounded set. In the nuclear space E, for any closed and absolutely convex neighborhood of the origin there exist a closed and absolutely convex bounded set such that the canonical mapping from the normed space $E(A)$ in the normed space $E(U)$ is nuclear. By Proposition 1, this canonical mapping is a precompact. Therefore, if we denote

$$A(U)=\{x(U)\in E(U)\,;\,x\in A\},$$

then $A(U)$ is precompact. There exists finite elements $x_1, x_2, \cdots, x_s$ of E such that

$$A(U)\subseteqq\bigcup_{n=1}^{S}\{x_n(U)+U(U)\},\quad U(U)=\{x(U)\in E(U)\,;\,x\in U\}.$$

This implies $A\subseteqq\bigcup_{n=1}^{S}\{x_n+U\}$. Therefore A is a precompact subset.

Definition 2. *Let E be a locally convex vector space. Let $\mathscr{B}$ denote a collection of u-compact subset E. The k-extension $k(u,\mathscr{B})$ of the topology u is defined as follows; A set V is $k(u,\mathscr{B})$-open if and only if $V\cap C$ is open in the relative u-topology of C for each C in $\mathscr{B}$.*

Proposition 3. *A nuclear space E is a S-space if on its dual E', $ew^*=\beta$.*

Proof. By Proposition 2, and the hypothesis, $\beta=p=ew^*$ on E'. Therefore E is a S-space. The following Lemma is due to [3].

Lemma 1. *Let E_u be a metrizable topological vector and $\mathscr{B}$ the collection of all u-compact subset of E_u. Then $u=k(u,\mathscr{B})$ is a vector topology.*

Theorem 1. *Let E be a nuclear space with a countable fundamental system of u-bounded set. Then E is a S-space.*

Proof. By Definition 1. (1) and Definition 2, $ew^*=k(\sigma,\mathscr{B}')$ $=k(\beta,\mathscr{B}')$, where $\mathscr{B}'$ is the collection of all $\sigma(E',E)$-closed convex equicontinuous sets of E'. Let $\mathscr{B}$ denote the class of all β-compact convex sets in E'. Each β-compact convex set is $\sigma(E',E)$-compact, then $k(\beta,\mathscr{B})\succ k(\beta,\mathscr{B}')$. Therefore we have

$$k(\beta,\mathscr{B})\succ k(\beta,\mathscr{B}')=ew^*\succ\beta.$$

But E_u contains a countable fundamental system of u-bounded subsets by hypothesis, hence β is metrizable, by Lemma 1,

$$\beta=k(\beta,\mathscr{B})=ew^*.$$

This shows that E_u is a S-space by proposition 3.

Since a complete S-space is B-complete [3], we have the following

Corollary. *The complete nuclear space with a countable fundamental system of u-bounded set is B-complete.*

Definition 3. *Let E be a locally convex space and E' its dual.*

(1) *If only all countable strong bounded subset of E are equi-*

continuous, then E is called a σ-quasibarrelled.

(2) *Let E be a σ-quasibarrelled, if there exists a countable fundamental system of bounded subset in E, then E is called a dualmetric.*

The following Lemma is due to [5].

Lemma 2. *Let E be a metrizable locally convex space and E' its dual. Then E', endowed with any locally convex topology finer than $\sigma(E', E)$ and coarser than $\tau(E', E)$, is a $B(\mathfrak{T})$-space.*

In the second paper [2], we shall prove that any nuclear dualmetric space which is quasicomplete is a Mackey space.

Theorem 2. *Let E be a nuclear dualmetric space which is quasicomplete, then E is a $B(\mathfrak{T})$-space.*

Proof. By Mackey's theorem, bounded sets under all locally convex topologies in the band $\tau(E, E') > u > \sigma(E, E')$ are the same. Since E is a nuclear space, an arbitrary bounded subset is a precompact by Proposition 2. On the other hand, E is a quasicomplete, therefore, $\sigma(E, E')$-bounded sets are relatively $\sigma(E, E')$-compact and $\beta = \tau(E', E)$ on E'. Clearly $E'^{\beta'} = E'^{\tau'} = E$ where $\beta = \beta(E', E)$, $\tau = \tau(E', E)$ and E'^{β} is metrizable by hypothesis. Moreover, E is a Mackey space (for detail see [2]). Hence by Lemma 2, E is a $B(\mathfrak{T})$-space.

J. L. Kelley has studied hypercomplete spaces. He proved that **a topological vector space E is hypercomplete if and only if each ew^*-closed convex circled subset of E' is $\sigma(E', E)$-closed.** On the other hand T. Husain [3] proved the following theorem. **Let E be a complete S-space. For a convex set M' in E' to be $\sigma(E', E)$-closed it is necessary and sufficient that $M' \cap U^0$ be $\sigma(E', E)$-closed for each neighborhood U of 0 in E.** Therefore, it is clear that a complete S-space is hypercomplete. Thus we have

Corollary. *A nuclear dualmetric space E which is quasicomplete is hypercomplete space.*

2. The purpose of this section is to study that the open mapping and closed graph theorems on a nuclear dualmetric space. By using the result Theorem 2 of Section 1, we have immediately the following

Theorem 3. *Let F be a barrelled space, and E a nuclear dualmetric space which is quasicomplete, then*

(a) *A linear and continuous mapping f of E onto F is open.*

(b) *A linear mapping g of F into E with the closed graph is continuous.*

Remark. A linear mapping f of dualmetric space into any locally convex space is continuous if and only if its restriction of the fundamental system of bounded set on B_n, is continuous (see [6, p. 401]). Dualmetric space is a (DF)-space (see [2]).

Moreover, we can prove the following

Theorem 4. *Let E be a nuclear dualmetric space and F a locally convex vector space with Baire's property, then*

(a) *Each linear mapping of E onto F is almost open.*

(b) *Each linear mapping of F into E is almost continuous.*

Proof. (a) Since E is a dualmetric, there is a countable fundamental system of bounded subset $B(E)=\{B_n\,;n=1,2,\cdots\}$ in E. Also E is a dualnuclear (see [8]), i.e., for arbitrary subset $B_n \in B(E)$ exist $B_m \in B(E)$ with $B_n \prec B_m$($\prec$; see [8]), and the canonical mapping; $B_n \to B_m$ is nuclear. Each nuclear mapping is precompact, so B_n is precompact and separable subset of normed space $E(B_m)$, where $E(B_m) =\{x \in E\,;x \in \rho B_m$ for every $\rho>0\}$. We can define Norm on $E(B_m)$ by $P_{B_m}(x)=\inf\{\rho>0\,;x\in\rho B\}$ for $x \in E(B_m)$. Moreover the identical mapping from $E(B_n)$ to E is continuous, therefore B_n is separable subset of E. We can choose double sequence $\{x_{m,n}\}$ which is dence in E. Now we replace the double sequence $\{x_{m,n}\}$ with $\{x_n\}_{n\in N}$. Let U and V be circled neighborhood of 0 in E such that $V+V\subset U$. Then clearly

$$\bigcup_{n\geq 1}\{x_n+V\}=E \quad \text{and} \quad F=f(E)=\bigcup_{n\geq 1}\{f(x_n+V)=\bigcup_{n\geq 1}\{f(x_n)+f(V)\}.$$

Since F has a Baire's property, at least for one n, $\overline{f(x_n)+f(V)}=f(x_n)+\overline{f(V)}$ has an interior point. Since $\overline{f(V)}$ and $f(x_n)+\overline{f(V)}$ are homeomorphic, $\overline{f(V)}$ has an interior point y. Since V is a circled neighborhood, $\overline{f(V)}$ is a circled, thus, $0=y-y$ is an interior point of

$$\overline{f(V)}+\overline{f(V)}\subset\overline{f(V)+f(V)}=\overline{f(V+V)}\subset\overline{f(U)}.$$

Hence, f is almost open mapping.

(b) This part follows clearly from (a).

Combining this theorem with Corollary of Theorem 1, we have the following

Corollary. *Let E be a nuclear dualmetric space which is quasi-complete and F a locally convex vector space with Baire's property, then each Linear continuous mapping E onto F is open.*

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