74. On Characterization of Regular Semigroups

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Let S be a semigroup,[1] and a be an arbitrary element of S. The principal bi-ideal of S generated by a is

(1) $$(a)_{(1,1)}=a\cup a^2\cup aSa.$$

If S is a regular semigroup, then by Theorem 7 in [4] every bi-ideal of S is of the form RL, where R is a right ideal, and L is a left ideal of S. Thus the product $(a)_R(a)_L=aSa$ is a bi-ideal of S, and it is easy to see that this is the least bi-ideal of S containing the element a. We show that the converse statement is also true, that is, if S is a semigroup such that the principal bi-ideal of S generated by a is aSa for each element a in S, then S is a regular semigroup. Since

(2) $$a\in(a)_{(1,1)}=aSa,$$

it follows that there exists at least one element x in S such that $a=axa$, i.e. S is a regular semigroup.

Thus we proved the following result.

Theorem 1. *A semigroup S is regular if and only if for each element a in S the principal bi-ideal of S generated by a is aSa.*

Similarly can be proved the following criterion, too.

Theorem 2. *A semigroup S is regular if and only if*

(3) $$(a)_{(1,1)}=(a)_R(a)_L$$

for each element a of S.

Proof. If S is a regular semigroup, then it is easy to show that the relation (3) holds.

Conversely, suppose that S is a semigroup having the property (3) for every element a in S. Then we have

(4) $$a\in(a)_{(1,1)}=(a)_R(a)_L,$$

and hence

(5) $$a\in(a\cup aS)(a\cup Sa)=a^2\cup aSa.$$

This means that either $a=a^2$ or $a\in aSa$. Therefore a is a regular element of S in both cases.

Theorem 1 in author's paper [3] and Theorem 1, Theorem 2 of this note imply the following result.

1) We adopt the terminology of Clifford and Preston [1]. See also Ljapin [5]. For other characterizations of regular semigroups we refer to Iséki [2] and Lajos [3].

Theorem 3. *For a semigroup S the following conditions are equivalent:*

(1) *S is regular.*

(2) *$R \cap L = RL$ for every right ideal R and left ideal L of S.*

(3) *$(a)_R \cap (b)_L = (a)_R(b)_L$ for every pair of elements a, b in S.*

(4) *$(a)_R \cap (a)_L = (a)_R(a)_L$ for each element a in S.*

(5) *$(a)_{(1,1)} = (a)_R(a)_L$ for each element a of S.*

(6) *$(a)_{(1,1)} = aSa$ for every element a in S.*

References

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