71. *Calculus in Ranked Vector Spaces. IV*

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1.9. The special case. (1.9.1) Proposition. *Let E be a normed vector space, $\{x_n\}$ a sequence of E and $x \in E$. Then for a sequence $\{x_n\}$ converges to x in the sense of ranked vector space it is necessary and sufficient that it converges to x in the sense of norm, i.e.,*

$$\{\lim x_n\} \ni x \Longleftrightarrow \lim \|x_n - x\| = 0.$$

Proof. (a) Suppose that $\{\lim x_n\} \ni x$, i.e., there exists a sequence $\{U_n(x)\}$ of neighborhoods of the point x and a sequence $\{\alpha_n\}$ of integers such that,

$$U_0(x) \supset U_1(x) \supset U_2(x) \supset \cdots \supset U_n(x) \supset \cdots, \quad 0 \le n < \omega_0,$$
$$\alpha_0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le \cdots, \quad 0 \le n < \omega_0,$$
$$\sup_n \alpha_n = \omega_0, \ U_n(x) \ni x_n, \text{ and } U_n(x) \in \mathfrak{V}_{\alpha_n},$$

for $n = 0, 1, 2, \cdots$.

By (1.6.6), each $U_n(x)$ is written in the following form, using $U_n(x) \in \mathfrak{V}_{\alpha_n}$,

$$U_n(x) = x + V_{\alpha_n}(0), \qquad n = 0, 1, 2, \cdots$$

where $V_{\alpha_n}(0) = \left\{x; \|x\| < \frac{1}{\alpha_n}\right\}$.

For every $\varepsilon > 0$, there exists a positive number N, using $\sup \alpha_n = \omega_0$, such that

$$n \ge N \Rightarrow \frac{1}{\alpha_n} < \varepsilon.$$

Since $U_n(x) = x + V_{\alpha_n}(0) \ni x_n$, $V_{\alpha_n}(0) \ni x_n - x$

$$\therefore \quad \|x_n - x\| < \frac{1}{\alpha_n}.$$

Thus if $n \ge N$, then

$$\|x_n - x\| < \frac{1}{\alpha_n} < \varepsilon$$
$$\therefore \quad \lim \|x_n - x\| = 0.$$

(b) Suppose coversely that $\lim \|x_n - x\| = 0$, then, for 1, there exists a positive number n_1 such that

$$n \ge n_1 \Rightarrow \|x_n - x\| < 1,$$
$$\therefore \quad V_1(0) \ni x_{n_1} - x, \ x_{n_1+1} - x, \ \cdots, \ x_{n_1+i} - x, \ \cdots$$

for $\frac{1}{2}$, there exists a positive number n_2 $(>n_1)$ such that

$$n \geq n_2 \Rightarrow ||x_n - x|| < \frac{1}{2},$$

$$\therefore \quad V_2(0) \ni x_{n_2} - x,\ x_{n_2+1} - x,\ \cdots,\ x_{n_2+i} - x,\ \cdots$$

....

for $\frac{1}{m}$, there exists a positive number n_m $(>n_{m-1})$ such that

$$n \geq n_m \Rightarrow ||x_n - x|| < \frac{1}{m},$$

$$\therefore \quad V_m(0) \ni x_{n_m} - x,\ x_{n_m+1} - x,\ \cdots,\ x_{n_m+i} - x,\ \cdots.$$

....

Let $V_m(0) + x = U_m(x)$ for $m = 0, 1, 2, \cdots$, then we have a sequence $\{U_m(x)\}$ of neighborhoods of the point x such that

$$U_1(x) \supset U_2(x) \supset U_3(x) \supset \cdots \supset U_m(x) \supset \cdots,$$

$$U_m(x) \ni x_{n_m},\ U_m(x) \in \mathfrak{V}_m, \text{ and } n_1 < n_2 < n_3 < \cdots < n_m < \cdots.$$

Now we define a sequence $\{U'_n(x)\}$ by

$U'_{n_1}(x) = U_1(x)$	$\ni x_{n_1}$	$U'_{n_1}(x) \in \mathfrak{V}_1$	$\alpha_{n_1} = 1$
$U'_{n_1+1}(x) = U_1(x)$	$\ni x_{n_1+1}$	$U'_{n_1+1}(x) \in \mathfrak{V}_1$	$\alpha_{n_1+1} = 1$
....	....	....	
$U'_{n_2-1}(x) = U_1(x)$	$\ni x_{n_2-1}$	$U'_{n_2-1}(x) \in \mathfrak{V}_1$	$\alpha_{n_2-1} = 1$
$U'_{n_2}(x) = U_2(x)$	$\ni x_{n_2}$	$U'_{n_2}(x) \in \mathfrak{V}_2$	$\alpha_{n_2} = 2$
....	....	....	
$U'_{n_3}(x) = U_3(x)$	$\ni x_{n_3}$	$U'_{n_3}(x) \in \mathfrak{V}_3$	$\alpha_{n_3} = 3.$
....	....	....	

Thus we obtain a sequence $\{U'_n(x)\}$ of neighborhoods of the point x and a sequence $\{\alpha_n\}$ of integers such that

$$U'_{n_1}(x) \supset U'_{n_1+1}(x) \supset U'_{n_1+2} \supset \cdots \supset U'_{n_1+i}(x) \supset \cdots$$

$$\alpha_{n_1} \leq \alpha_{n_1+1} \leq \alpha_{n_1+2} \leq \cdots \leq \alpha_{n_1+i} \leq \cdots$$

$$U_{n_1+i}(x) \ni x_{n_1+i},\ \sup \alpha_{n_1+i} = \omega_0, \text{ and } U_{n_1+i}(x) \in \mathfrak{V}_{\alpha_{n_1+i}}.$$

$$\therefore \quad \{\lim_i x_{n_1+i}\} \ni x.$$

By (1.2.3) we have

$$\{\lim_n x_n\} \ni x.$$

(1.9.2) **Proposition.** *Let E be a normed vector space and $\{x_n\}$ a sequence of points in E. Then for a sequence $\{x_n\}$ in E to be a quasi-bounded sequence it is necessary and sufficient that the sequence $\{||x_n||\}$ is bounded.*

Proof. (a) Suppose that $\{x_n\}$ is a quasi-bounded sequence. If our assertion were false, then there would exists a subsequence $\{x_{n_i}\}$ such that

$$||x_{n_0}|| < ||x_{n_1}|| < ||x_{n_2}|| < \cdots < ||x_{n_i}|| < \cdots$$

and

$$\lim_{i \to \infty} ||x_{n_i}|| = \infty.$$

Then
$$\frac{1}{\sqrt{||x_{n_i}||}}\to 0 \qquad \text{for } i\to\infty,$$
and
$$\left\|\frac{1}{\sqrt{||x_{n_i}||}}x_{n_i}\right\|=\sqrt{||x_{n_i}||}\to\infty \qquad \text{for } i\to\infty.$$
This contradicts that by (1.7.4) $\{x_{n_i}\}$ is a quasi-bounded sequence.

Therefore $\{||x_n||\}$ is bounded.

(b) Suppose conversely that $\{||x_n||\}$ is bounded, i.e., there exists a number M such that
$$||x_n||<M, \qquad n=0, 1, 2, \cdots.$$
Let $\{\mu_n\}$ be a sequence in $\mathfrak{R}$ with $\mu_n\to 0$, then we have
$$0\leq ||\mu_n x_n||<|\mu_n|M.$$
Since $|\mu_n|M\to 0$ for $n\to\infty$,
$$||\mu_n x_n||\to 0 \qquad \text{for } n\to\infty.$$
That is, $\{x_n\}$ is a quasi-bounded sequence.

(1.9.3) **Proposition.** *If E is a normed vector space, then it is a separated ranked vector space.*

Proof. It suffices to show that E satisfies (1.4.1) axiom (T_0). For this let x, y be arbitrary elements in E and $x\neq y$. i.e.,
$$||x-y||=2a>0.$$
We can find a positive integer N such that $\frac{1}{N}<a$. Suppose that $m, n\geq N$, $U(x)\in\mathfrak{B}_m$, and $V(y)\in\mathfrak{B}_n$. If $x'\in U(x)$, since by (1.6.6) $U(x)=x+V_m(0)$, it can be written in the following way:
$$x'=x+x_1,$$
where $x_1\in V_m(0)$. Using $V_m(0)=\left\{x\,;\, ||x||<\frac{1}{m}\right\}$, we have
$$||x'-x||=||x_1||<\frac{1}{m}$$
$$\therefore\quad ||x'-x||<a.$$
Analogously, if $y'\in V(y)$, then
$$||y'-y||<a.$$
Now
$$||x'-y||=||x'-x+x-y||\geq ||x-y||-||x'-x||>a$$
$$\therefore\quad ||x'-y||>a \qquad \therefore\quad x'\notin V(y)$$
$$\therefore\quad U(x)\cap V(y)=\phi.$$
That is, the axiom (T_0) holds in E, and therefore E is a separated ranked vector space.

(1.9.4) **Proposition.** *Let E be a normed vector space, $\{x_n\}$ a sequence in E and $x\in E$. Then for $\{x_n\}$ converges to x in the sense of ranked vector space it is necessary and sufficient that $\{x_n\}$ converges to x in the sense of L-convergence, i.e.,*
$$\{\lim x_n\}\ni x \Longleftrightarrow \{\mathrm{Lim}\, x_n\}\ni x.$$

Proof. By (1.8.3), it suffices to prove that $\{\lim x_n\} \ni x$ implies $\{\mathrm{Lim}\, x_n\} \ni x$.

Suppose that $\{\lim x_n\} \ni x$. By (1.9.1), we have

$$\lim \|x_n - x\| = 0.$$

Each $x_n - x$ can be represented in the following form:

$$x_n - x = \|x_n - x\| \frac{x_n - x}{\|x_n - x\|},$$

where $\|x_n - x\| \to 0$ and, since $\left\| \frac{x_n - x}{\|x_n - x\|} \right\| = 1$ for $n = 0, 1, 2, \cdots$, $\left\{ \frac{x_n - x}{\|x_n - x\|} \right\}$ is a quasi-bounded sequence in E.

$$\therefore \quad \{\mathrm{Lim}\,(x_n - x)\} \ni 0.$$

$$\therefore \quad \{\mathrm{Lim}\, x_n\} \ni x.$$

§ 2. Differentiability and derivatives. In this section, the definition of differentiability is given and the most elementary results of calculus are proved.

2.1. Remainder. (2.1.1) **Definition.** Let $r: E_1 \to E_2$ be a map between ranked vector spaces E_1, E_2. Then we associate to r a new map $\theta_r: \Re \times E_1 \to E_2$ defined by

$$\begin{aligned} \theta_r(\lambda, x) &= \frac{r(\lambda x)}{\lambda} && \text{if } \lambda \neq 0 \\ &= 0 && \text{if } \lambda = 0. \end{aligned}$$

(2.1.2) **Definition.** A map $r: E_1 \to E_2$ is called a *remainder*, and we write $r \in R(E_1; E_2)$ if and only if

(1) $r(0) = 0$,

(2) for any quasi-bounded sequence $\{x_n\}$ and for a sequence $\{\lambda_n\}$ in $\Re$ such that $\lambda_n \to 0$,

$$\{\lim \theta_r(\lambda_n, x_n)\} \ni 0.$$

Example. The zero map is a remainder.

(2.1.3) **Proposition.** *If $r: E_1 \to E_2$ is a remainder, then it is continuous at the point zero in the sense of L-convergence.*

Proof. Let $\{\mathrm{Lim}\, x_n\} \ni 0$, i.e.,

$$x_n = \lambda_n x'_n \qquad n = 0, 1, 2, \cdots$$

where $\lambda_n \to 0$ in $\Re$ and $\{x'_n\}$ is a quasi-bounded sequence in E.

$$r(x_n) = r(\lambda_n x'_n) = \lambda_n \frac{r(\lambda_n x'_n)}{\lambda_n}.$$

By assumption one has

$$\left\{ \lim \frac{r(\lambda_n x'_n)}{\lambda_n} \right\} \ni 0,$$

and so by (1.7.2) $\left\{ \frac{r(\lambda_n x'_n)}{\lambda_n} \right\}$ is a quasi-bounded sequence.

$$\therefore \quad \{\mathrm{Lim}\, r(x_n)\} \ni 0.$$

That is, $r: E_1 \to E_2$ is continuous at the point zero in the sense of L-convergence.

(2.1.4) **Proposition.** *$R(E_1; E_2)$ is a vector space, i.e., for any $r_1, r_2 \in R(E_1; E_2)$ and for any $\alpha_1, \alpha_2 \in \mathfrak{R}$,*

$$\alpha_1 r_1 + \alpha_2 r_2 \in R(E_1; E_2).$$

Proof. (1) $(\alpha_1 r_1 + \alpha_2 r_2)(0) = \alpha_1(r_1(0)) + \alpha_2(r_2(0)) = 0.$

(2) Let $\{x_n\}$ be a quasi-bounded sequence and $\{\lambda_n\}$ a sequence in $\mathfrak{R}$ such that $\lambda_n \to 0$, then

$$\begin{aligned} \theta_{\alpha_1 r_1 + \alpha_2 r_2}(\lambda_n, x_n) &= \frac{(\alpha_1 r_1 + \alpha_2 r_2)(\lambda_n x_n)}{\lambda_n} \\ &= \frac{\alpha_1(r_1(\lambda_n x_n)) + \alpha_2(r_2(\lambda_n x_n))}{\lambda_n} \\ &= \alpha_1 \frac{r_1(\lambda_n x_n)}{\lambda_n} + \alpha_2 \frac{r_2(\lambda_n x_n)}{\lambda_n}. \end{aligned}$$

Since r_1, r_2 are remainders,

$$\left\{\lim \frac{r_1(\lambda_n x_n)}{\lambda_n}\right\} \ni 0, \quad \text{and} \quad \left\{\lim \frac{r_2(\lambda_n x_n)}{\lambda_n}\right\} \ni 0.$$

$$\therefore \quad \{\lim \theta_{\alpha_1 r_1 + \alpha_2 r_2}(\lambda_n, x_n)\} \ni 0.$$

$$\therefore \quad \alpha_1 r_1 + \alpha_2 r_2 \in R(E_1; E_2).$$

Thus $R(E_1; E_2)$ is a vector space.

(2.1.5) **Definition.** We denote by $L(E_1; E_2)$ the set of all linear and continuous maps between ranked vector spaces E_1, E_2. Then $L(E_1, E_2)$ is also a vector space. Indeed if $l_1, l_2 \in L(E_1; E_2)$ and $\alpha_1, \alpha_2 \in \mathfrak{R}$, it follows, using $l_1, l_2 \in L(E_1, E_2)$, that $\alpha_1 l_1 + \alpha_2 l_2$ is also linear and continuous, i.e., $\alpha_1 l_1 + \alpha_2 l_2 \in L(E_1; E_2)$.

Example. The zero map belongs to $L(E_1; E_2)$.

(2.1.6) **Lemma.** *If $r \in R(E_1; E_2)$ and $l \in L(E_2; E_3)$, then*

$$l \cdot r \in R(E_1; E_3).$$

Proof. (1) $(l \cdot r)(0) = l(r(0)) = l(0) = 0.$

(2) Let $\{x_n\}$ be a quasi-bounded sequence in E, and $\{\lambda_n\}$ a sequence in $\mathfrak{R}$ such that $\lambda_n \to 0$, then

$$\theta_{l \cdot r}(\lambda_n, x_n) = \frac{(l \cdot r)(\lambda_n x_n)}{\lambda_n} = \frac{l(r(\lambda_n x_n))}{\lambda_n}$$

since $l: E_1 \to E_2$ is linear,

$$= l\left\{\frac{r(\lambda_n x_n)}{\lambda_n}\right\}.$$

By assumption we have

$$\left\{\lim \frac{r(\lambda_n x_n)}{\lambda_n}\right\} \ni 0,$$

and since $l: E_1 \to E_2$ is continuous,

$$\left\{\lim l\left(\frac{r(\lambda_n x_n)}{\lambda_n}\right)\right\} \ni l(0)=0$$

$$\therefore \quad \{\lim \theta_{l\cdot r}(\lambda_n x_n)\} \ni 0$$

$$\therefore \quad l\cdot r \in R(E_1\,;E_3).$$

(2.1.7) **Lemma.** *Let* $r \in R(E_1\,;E_2)$, $l \in L(E_1\,;E_2)$, *and* $r' \in R(E_2\,;E_3)$, *then*

$$r'\cdot(l+r) \in R(E_1\,;E_3).$$

Proof. (1) $(r'(l+r))(0)=r'(l(0)+r(0))=r'(0)=0$.

(2) Let $\{x_n\}$ be a quasi-bounded sequence in E_1 and $\{\lambda_n\}$ a sequence in $\mathfrak{R}$ such that $\lambda_n \rightarrow 0$, then

$$\theta_{r'\cdot(l+r)}(\lambda_n, x_n)=\frac{1}{\lambda_n}(r'\cdot(l+r))(\lambda_n x_n)$$

$$=\frac{1}{\lambda_n}[r'(l(\lambda_n x_n)+r(\lambda_n x_n))].$$

Since l is linear,

$$=\frac{1}{\lambda_n}[r'(\lambda_n l(x_n)+r(\lambda_n x_n)]$$

$$=\frac{1}{\lambda_n}\left[r'\left\{\lambda_n\left(l(x_n)+\frac{r(\lambda_n x_n)}{\lambda_n}\right)\right\}\right].$$

By assumption we have

$$\left\{\lim \frac{r(\lambda_n x_n)}{\lambda_n}\right\} \ni 0.$$

Hence $\left\{\frac{r(\lambda_n x_n)}{\lambda_n}\right\}$ is a quasi-bounded sequence. By (1.7.7) $\{l(x_n)\}$ is also a quasi-bounded sequence. Therefore it follows from (1.7.5) that $\left\{l(x_n)+\frac{r(\lambda_n x_n)}{\lambda_n}\right\}$ is a quasi-bounded sequence. Thus, since $r' \in R(E_2\,;E_3)$,

$$\left\{\lim\left[\frac{1}{\lambda_n}r'\left(\lambda_n\left(l(x_n)+\frac{r(\lambda_n x_n)}{\lambda_n}\right)\right)\right]\right\} \ni 0$$

$$\therefore \quad r'\cdot(l+r) \in R(E_1\,;E_3).$$