67. Characterizations of Self-Injective Rings

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(Comm. by Kenjiro SHODA, M. J. A., May 13, 1968)

In the theory of (non-commutative) rings, self-injective rings are one of the most attractive objects, and have been studied in the last two decades by many authors. It is well known that a ring R with identity element is right self-injective if and only if, for each right ideal I and for each map $f: I_R \to R_R$, there exists $a \in R$ such that $f(i) = ai$ for all $i \in I$ (See Baer [1, Theorem 1]). The theory of QF-rings provides us with many characterizations of self-injective rings with minimum condition. For example, the following conditions are equivalent for a (left or right) Artinian ring R:

(1) *R is right self-injective.*

(2) *$l(r(L))=L$, $r(l(I))=I$ for each left ideal L and right ideal I.*

(3) *If aR (resp. Ra), $a \in R$, is simple then $l(r(a))=Ra$ (resp. $r(l(a)) = aR$).*

For a discussion of the condition (3), see Kato [6, Lemma 2].

In this paper we shall give some characterizations of right self-injective rings in terms of duality.

1. Preliminaries. Throughout this paper each ring R will be a ring with identity element and each module over R will be unital.

If A is a right R-module, let $A^* = \mathrm{Hom}_R(A, R)$ be its dual and let $\delta_A : A \to A^{**}$ be the natural map. We call, as usual, A torsionless (resp. reflexive) if δ_A is a monomorphism (resp. an isomorphism). If X is a subset of A (resp. A^*), then we set

$$l(X) = \{b \in A^* \mid bX = 0\} \text{ (resp. } r(X) = \{a \in A \mid Xa = 0\}).$$

We shall have need of the following lemma for our characterizations of right self-injective rings.

Lemma 1. **(Rosenberg and Zelinsky [7, Theorem 1.1]).** *Let R be a right self-injective ring, A a right R-module, and B a finitely generated submodule of A^*. Then $l(r(B))=B$.*

Proof. Write $B = Rb_1 + \cdots + Rb_n$, $b_i \in B$, and let $b \in l(r(B))$. Then $\bigcap_{i=1}^{n} r(b_i) = r(B) \subset r(b)$. Hence there exists a map $f: \overset{n}{\oplus} R_R \to R_R$ such that $(b_1 a, \cdots, b_n a) \to ba$, $a \in A$, by virtue of the injectivity of R_R. Then

$$ba = f(b_1 a, \cdots, b_n a) = f(b_1 a, 0, \cdots, 0) + \cdots + f(0, \cdots, 0, b_n a)$$
$$= r_1 b_1 a + \cdots + r_n b_n a,$$

where $f(\cdots 0\cdots, b_i a, \cdots 0\cdots)=r_i b_i a$ for some $r_i \in R$, making use of the injectivity of R_R. Hence $b=r_1 b_1+\cdots+r_n b_n \in B$, which proves $l(r(B))=B$.

2. Characterizations of right self-injective rings. Let A be a right R-module, A_0 a submodule of A. We denote by $A\ '\supset A_0$ the fact that A is an essential extension of A_0 and by $E(A)$ the injective hull of A (see Eckmann and Schopf [3]). It is well known that $E(A)$ always exists uniquely up to isomorphism over A (see [3]). Needless to say, $E(A)$ is injective and $E(A)\ '\supset A$.

Proposition 1. *The following conditions are equivalent for any ring R:*

(1) *R is right self-injective.*

(2) *$E(R_R)$ is projective and $r(L)\neq 0$ for each finitely generated left ideal $L\neq R$.*

Proof. (1) implies (2). Let R be right self-injective, L a finitely generated proper left ideal. Then $l(r(L))=L\neq R$ by the above lemma, consequently $r(L)\neq 0$. The projectivity of $E(R_R)$ is obvious since $E(R_R)=R$.

(2) implies (1). Assume (2). Then R is a direct summand of $E(R_R)$ by Bass [2, Theorem 5.4]. Thus R is right self-injective.

Remark. If $E(R_R)$ is torsionless and R is a right S-ring (that is, $r(L)\neq 0$ for each left ideal $L\neq R$), then R is right self-injective by [6, Lemma 1].

If $\alpha \in E(R_R)$, then we set

$$(R:\alpha)=\{a\in R \mid \alpha a \in R\}.$$

The following characterization of right self-injective rings is essentially due to Ikeda and Nakayama [4, Theorem 1].

Proposition 2. *The following conditions are equivalent for any ring R:*

(1) *R is right self-injective.*

(2) *$l(r(a))=Ra$ for each $a\in R$, and $l(I_1\cap I_2)=l(I_1)+l(I_2)$ for finitely generated right ideals I_1, I_2. Moreover $(R:\alpha)$ is a finitely generated right ideal for each $\alpha\in E(R_R)$.*

Proof. (1) implies (2). The first part of the statement (2) follows from [4, Theorem 1]. Next, since R is right self-injective, $(R:\alpha)=R$ for each $\alpha \in E(R_R)=R$.

(2) implies (1). Note that the first part of the condition (2) is equivalent to the vanishing of $\mathrm{Ext}^1_R(R/I, R)$ for each finitely generated right ideal I by [4, Theorem 1]. Now we must show that $E(R_R)=R$. Assume that there exist $\alpha\in E(R_R)$ such that $\alpha\notin R$. Since

$$R+\alpha R/R \approx R/(R:\alpha),$$

and $(R:\alpha)$ is finitely generated by assumption, we have

$\operatorname{Ext}^1_R(R+\alpha R/R, R)=0$. It follows from this that R is a direct summand of $R+\alpha R$. This contradicts the fact that $R+\alpha R\ '\supset R$, $R+\alpha R \neq R$. Therefore $E(R_R)=R$.

Remark. $R_R{}'\supset(R:\alpha)$ since $\alpha R\ '\supset R\cap\alpha R$.

We are now in a position to prove our main theorem.

Theorem 1. *The following conditions on a ring R are equivalent:*

(1) *R is right self-injective.*

(2) *$\delta_I: I\to I^{**}$ is an essential monomorphism (that is, $I^{**}{}'\supset Im\delta_I$) for each right ideal I. Moreover, for each right R-module A and each cyclic proper submodule B of A^*, $(A^*/B)^*\neq 0$.*

Proof. (1) implies (2). Let R be a right self-injective ring and I a right ideal. Then $r(l(I))\ '\supset I$. In fact, write $E(I)=eR$, $e=e^2\in R$, making use of the right self-injectivity of R. Then

$$E(I)=eR=r(l(e))\supset r(l(I))\supset I.$$

Thus $r(l(I))\ '\supset I$ since $E(I)\ '\supset I$. Next, since $\operatorname{Ext}^1_R(R/I, R)=0$, $I^{**}\approx(R/l(I))^*\approx r(l(I))$ (see Kato [5, Proposition 5]) and we have the following commutative diagram with exact rows

$$\begin{array}{ccccc} 0\to & Im\delta_I & \to & I^{**} & \\ & \delta_I\ \wr\wr & & \wr\wr & \\ 0\to & I & \to & r(l(I)). & \end{array}$$

Consequently $I^{**}{}'\supset Im\delta_I$ since $r(l(I))\ '\supset I$. Now, let A be a right R-module, B a cyclic proper submodule of A^*. Let us show that $(A^*/B)^*\neq 0$. By Lemma 1, $r(B)\supset r(A^*)$, $r(B)\neq r(A^*)$. Hence we can choose $a\in r(B)$ such that $a\notin r(A^*)$. It is then easy to see that $\delta_A(a)$ induces a nonzero map $A^*/B\to {}_RR$, or equivalently, $(A^*/B)^*\neq 0$.

(2) implies (1). Assume (2) and let I be a right ideal. We must show that $\operatorname{Ext}^1_R(R/I, R)=0$. We have the dual exact sequence

$$(R_R)^*\to I^*\to\operatorname{Ext}^1_R(R/I, R)\to 0.$$

Notice that $\operatorname{Ext}^1_R(R/I, R)\approx I^*/B$ for some cyclic submodule B of I^*. Now dualize the above exact sequence to get the following commutative diagram with exact rows

$$\begin{array}{ccccc} 0\to\operatorname{Ext}^1_R(R/I, R)^*\to & I^{**} & \to & (R_R)^{**} \\ & \delta_I\uparrow & & \wr\wr \\ 0\longrightarrow & I & \longrightarrow & R_R. \end{array}$$

Therefore the composition map $I\to I^{**}\to(R_R)^{**}$ is a monomorphism. But this implies, by the assumption that $\delta_I: I\to I^{**}$ is essential, $I^{**}\to(R_R)^{**}$ must be a monomorphism. Hence $\operatorname{Ext}^1_R(R/I, R)^*=0$ by the exactness of the above diagram. Since, as was seen, $\operatorname{Ext}^1_R(R/I, R)\approx I^*/B$ with cyclic B, it follows from our assumption that $\operatorname{Ext}^1_R(R/I, R)=0$. We have thus proved our theorem.

Remark. Let R be a right self-injective ring. Then each right ideal is reflexive if and only if R_R is a cogenerator in the category of right R-modules.

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