106. σ-Spaces and Closed Mappings. I

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1. Introduction. In our previous paper [7] we have introduced the notion of a σ-locally finite net as a generalization of a countable net (cf. [2], [5]) and studied the spaces with σ-locally finite nets as a class of a topological spaces which contains all metric spaces.

Definition. A collection $\mathfrak{B}$ of subsets of a topological space X is said to be a *net for* X if the following condition is satisfied:

For every point x of X and every open neighborhood U of x there exists an element B of $\mathfrak{B}$ with $x \in B \subset U$.

A collection $\mathfrak{B}$ of subsets of X is said to be a *σ-locally finite net* if it is a net and it is a union of a countable number of subcollections which are locally finite in X. We shall say that X is a *σ-space* if X has a σ-locally finite net (cf. [6]).

The notion of net was introduced and discussed by A. Arhangel'skii [1][1)] and several results were obtained by him in [1], [2] and, also, by E. Michael [4] in the case of countable nets.

The purpose of this paper is to study the images of σ-spaces under closed mappings[2)] and to prove the following two theorems.

Theorem 1. *Let f be a closed mapping of a normal T_1 σ-space X onto a topological space Y. Then the set $\{y \mid \partial f^{-1}(y)$ is not countably compact$\}$ is a σ-discrete subset of Y;* that is, *it is a countable union of discrete subsets of Y, where $\partial f^{-1}(y)$ denotes the boundary of $f^{-1}(y)$ for each $y \in Y$.*

Theorem 2. *Let f be a closed mapping of a normal T_1 σ-space X onto a topological space Y. Then Y is a σ-space, too.*

As regards Theorem 1 N. Lašnev [3] proved it in the case of metric space. He proved also, in another paper [4], the following theorem:

In order that a T_1 space X be a closed image of a metric space, it is necessary and sufficient that X is a Fréchet-Urysohn space[3)] with

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1) This fact was pointed out to us by Professor A. Arhangel'skii. We express our thanks to his advice.

2) All mappings in this paper are *continuous*.

3) X is a *Fréchet-Urysohn space* if, for every subset M of X and $x_0 \in \overline{M}$, there exists a sequence $\{x_n \mid n=1, 2, \cdots\}$ of points of M, converging to x_0.

an almost refining sequence of hereditarily conservative coverings[4] *comprising a net for* X.

By Theorem 2 we can slightly strengthen his result as below:

Corollary. *In order that a* T_1 *space* X *be a closed image of a metric space, it is necessary and sufficient that* X *is a Fréchet-Urysohn space with an almost refining sequence of locally finite coverings comprising a net for* X.

In § 2 we shall give the related observations which will be needed for the next section and prove our theorems in § 3.

2. Preliminaries. Lemma 1. *If* X *is a* σ*-space, then* X *has a* σ*-locally finite net* $\mathfrak{B}=\bigcup_{n=1}^{\infty}\mathfrak{B}_n$ *satisfying the following three conditions*:

(a) $\mathfrak{B}_n$ *is a locally finite covering of* X *for* $n=1,2,\cdots$,

(b) $\mathfrak{B}_n\subset\mathfrak{B}_{n+1}$ *for* $n=1,2,\cdots$,

(c) $\mathfrak{B}_n$ *is closed under finite intersections*; that is, $\mathfrak{B}_n$ *contains any of intersections of finite members of* $\mathfrak{B}_n$ *for* $n=1,2,\cdots$.

Proof. Let $\mathfrak{C}=\bigcup_{n=1}^{\infty}\mathfrak{C}_n$ be a given σ-locally finite net for X. Without loss of generality we can assume $X\in\mathfrak{C}_1$. Put $\mathfrak{D}_n=\bigcup_{i\leq n}\mathfrak{C}_i$ and let $\mathfrak{B}_n$ be the collection of all intersections of finite members of $\mathfrak{D}_n$ for $n=1,2,\cdots$. Then it is easily seen that $\mathfrak{B}=\bigcup_{n=1}^{\infty}\mathfrak{B}_n$ is a σ-locally finite net for X satisfying all conditions.

Definition. Let x be a point of a topological space X and $\{S_n(x)\,|\,n=1,2,\cdots\}$ a sequence of subsets of X with $x\in S_n(x)$ for $n=1,2,\cdots$. Then we shall say that $\{S_n(x)\,|\,n=1,2,\cdots\}$ is a *strict* x-*sequence* if any sequence $\{x_n\,|\,n=1,2,\cdots\}$ with $x_n\in S_n(x)$ converges to x. The notion of x-sequence was introduced by A. Arhangel'skii [2] in the sense that any sequence $\{x_n\,|\,n=1,2,\cdots\}$ has an accumulation point in X.

Proposition 1. *Let* X *be a* σ*-space. Then* X *has a* σ*-locally finite net* $\mathfrak{B}$ *such that for each point* $x\in X$ *there exists a subcollection* $\{S_n(x)\,|\,n=1,2,\cdots\}$ *of* $\mathfrak{B}$ *which is a strict* x*-sequence.*

Proof. Let $\mathfrak{B}=\bigcup_{n=1}^{\infty}\mathfrak{B}_n$ be a σ-locally finite net for X satisfying all conditions in Lemma 1. For an arbitrary point $x\in X$ select a minimal

4) A system $\{F_\alpha\,|\,\alpha\in\mathfrak{A}\}$ of closed subsets of X is said to be *hereditarily conservative* if for any subset $\mathfrak{A}'$ of $\mathfrak{A}$ and any system $\{M_\alpha\,|\,\alpha\in\mathfrak{A}'\}$ of closed subsets of X, such that $M_\alpha\subset F_\alpha$, the set $\bigcup_{\alpha\in A'}M_\alpha$ is closed in X. A sequence $\{\mathfrak{B}_i\,|\,i=1,2,\cdots\}$ of closed coverings of X is said to be *almost refining* if for any point $x_0\in X$, any system $\{Bi\,|\,i=1,2,\cdots\}$, such that $B_i\in\mathfrak{B}_i$ and $x_0\in B_i$, is either hereditarily conservative, or else forms a net at x_0.

member $S_n(x)$ of $\mathfrak{B}_n$ containing x for $n=1,2,\cdots$. The existence of such $S_n(x)$ is assured by (c) of Lemma 1. Since $\mathfrak{B}$ is a net for X, we can see that $\{S_n(x)\mid n=1,2,\cdots\}$ is a strict x-sequence. This completes the proof.

Definition. Let x be a point of topological space X and $\mathfrak{B}_x$ a collection of subsets of X with $x\in B$ for each $B\in\mathfrak{B}_x$. Then we shall say that $\mathfrak{B}_x$ is a *local net at* x if, whenever $x\in U$ with U open in X, then $x\in B\subset U$ for some $B\in\mathfrak{B}_x$ (cf. [8]).

From the definition we can obtain the following, immediately.

Proposition 2. *Let x be a point of a topological space X and $\{B_n\mid n=1,2,\cdots\}$ a decreasing sequence of subsets of X which is a local net at x. Then $\{B_n\mid n=1,2,\cdots\}$ is a strict x-sequence.*

Lemma 2. *Let $\mathfrak{B}'$ and $\mathfrak{C}$ be nets for T_2 spaces X and Y, respectively, and f a closed mapping of X onto Y. Then the collection $\mathfrak{B}=\{B'\cap f^{-1}(C)\mid B'\in\mathfrak{B}', C\in\mathfrak{C}\}$ is a net for X having the following property:*

If $\bigcap_{i=1}^{k} f(B_{\alpha_i})$ is finite subset of Y for $B_{\alpha_1},\cdots,B_{\alpha_k}\in\mathfrak{B}$, then for each $y\in\bigcap_{i=1}^{k} f(B_{\alpha_i})$ there exist $B_{\beta_1},\cdots,B_{\beta_l}$ in $\mathfrak{B}$ such that $\{y\}=\bigcap_{i=1}^{l} f(B_{\beta_i})$.

Proof. It is easily seen that $\mathfrak{B}$ is a net for X. Now let us put

$$\bigcap_{i=1}^{k} f(B_{\alpha_i})=\{y_1,\cdots,y_m\}\quad\text{for}\quad B_{\alpha_1},\cdots,B_{\alpha_k}\in\mathfrak{B}.$$

Since Y is a T_2, space, there exists a disjoint collection $\{V_1,\cdots,V_m\}$ of open subsets of Y such that $y_i\in V_i$ for $i=1,\cdots,m$. Since $\mathfrak{C}$ is a net for Y, there exist $C_1,\cdots,C_m$ in $\mathfrak{C}$ with $y_i\in C_i\subset U_i$ for $i=1,\cdots,m$. Hence, if we put $B_{ij}=f^{-1}(C_i)\cap B_{\alpha_j}$ for $j=1,\cdots,k$; $i=1,\cdots,m$, then each B_{ij} is a member of $\mathfrak{B}$ with $y_i\in f(B_{ij})$. Accordingly, we have $\bigcap_{j=1}^{k} f(B_{ij})=\bigcap_{j=1}^{k}[C_i\cap f(B_{\alpha_j})]=C_i\cap[\bigcap_{j=1}^{k} f(B_{\alpha_j})]=\{y_i\}$ for $i=1,\cdots,m$.

This completes the proof.

Proposition 3. *Let f be a closed mapping of a topological space X onto a topological space Y and $\mathfrak{F}$ an arbitrary locally finite closed covering of X. Then the set*

$$K=\{y\mid\{y\}=\bigcap_{i=1}^{n} f(F_{\alpha_i})\ \textit{for some } n \textit{ and for some } F_{\alpha_1},\cdots,F_{\alpha_n}\in\mathfrak{F}\}$$

is discrete in X.

Proof.[5] For an arbitrary point $y_0\in Y$ let

$$V=Y-\cup\{f(F_\alpha)\mid F_\alpha\in\mathfrak{F}, y_0\notin f(F_\alpha)\}.$$

Then V is an open subset of Y containing y_0 by the assumption. Now it is sufficient to show that V contains at most one point of K. If y

5) After we have proved this proposition, Professor J. Suzuki pointed out that it may be proved more shortly. Here we do it according to his suggestion.

is a point of $V \cap K$, we can see that $y \in K$ implies $\{y\}=\bigcap_{i=1}^{n} f(F_{\alpha_i})$ for some n and for some $F_{\alpha_1}, \cdots, F_{\alpha_n} \in \mathfrak{F}$, and that $y \in V$ implies $y_0 \in f(F_{\alpha_i})$ for $i=1, \cdots, n$. Hence we have $\{y\}=\bigcap_{i=1}^{n} f(F_{\alpha_i}) \ni y_0$ and, consequently, $y=y_0$. This completes our proof.

3. Proof of Theorems. Proof of Theorem 1. Let $\mathfrak{B}=\bigcup_{n=1}^{\infty} \mathfrak{B}_n$ be a σ-locally finite net for X satisfying all conditions in Lemma 1 and $\mathfrak{C}_n=f(\mathfrak{B}_n)=\{f(B) \mid B \in \mathfrak{B}_n\}$ for $n=1, 2, \cdots$. Then it is easily seen that $\mathfrak{C}=\bigcup_{n=1}^{\infty} \mathfrak{C}_n$ is a net Y. Furthermore, since X is a regular T_1 space, we can assume that each $B \in \mathfrak{B}$ is closed in X.

Now, let us put

$$Y_n=\{y \mid \{y\}=\bigcap_{i=1}^{k} f(B_{\alpha_i}) \text{ for some } k \text{ and for some } B_{\alpha_1}, \cdots, B_{\alpha_k} \in \mathfrak{B}_n\}$$

for $n=1, 2, \cdots$. Then each Y_n is discrete in Y by Proposition 3. Accordingly, it is sufficient to show that for each $y \in Y-\bigcup_{n=1}^{\infty} Y_n$, $\partial f^{-1}(y)$ is countably compact.

On the contrary, let us assume that $\partial f^{-1}(y_0)$ is not countably compact for some $y_0 \in Y-\bigcup_{n=1}^{\infty} Y_n$. Then $\partial f^{-1}(y_0)$ contains an infinite and discrete subset $\{x_n \mid n=1, 2, \cdots\}$ of X. Since X is a normal T_1 space, there exists a discrete collection $\{U_n \mid n=1, 2, \cdots\}$ of open subsets of X with $x_n \in U_n$ for $n=1, 2, \cdots$. For each n, let $B_i(x_n)$ be the minimal member in $\mathfrak{B}_i$ with $x_n \in B_i(x_n)$. Then it is easily seen that each sequence $\{B_i(x_n) \mid i=1, 2, \cdots\}$ is a decreasing, local net at x_n and each sequence $\{f(B_i(x_n) \mid i=1, 2, \cdots\}$ is a decreasing, local net at y_0, too. Hence we can choose an l_n with $x_n \in B_{l_n}(x_n) \subset U_n$ for each n and, without loss of generality, we can assume that $l_1 < l_2 < \cdots$. If we put $C_k=\bigcap_{n=1}^{k} fB_{l_k}(x_n))$ for each k, then $\{C_k \mid k=1, 2, \cdots\}$ is a decreasing sequence in Y, each of which contains y_0 and, in addition, it is a local net at y_0. Therefore, it is a strict y_0-sequence by Proposition 2. Since y_0 is not in $\bigcup_{n=1}^{\infty} Y_n$, each C_k is an infinite subset of Y by Lemma 2. Accordingly, we can select a point p_n in $B_{l_n}(x_n)$ for each n such that $\{f(p_n) \mid n=1, 2, \cdots\}$ is an infinite subset of Y with $f(p_n) \in C_n$ and $f(p_n) \neq y_0$ for each n. Consequently, we obtain a sequence $\{f(p_n) \mid n=1, 2, \cdots\}$ in Y converging to y_0. On the other hand, since $\{U_n \mid n=1, 2, \cdots\}$ is discrete, $\{p_n \mid n=1, 2, \cdots\}$ is also discrete in X and, since f is a closed mapping, $\{f(p_n) \mid n=1, 2, \cdots\}$ must be closed in Y. This is a contradiction. The proof is completed.

Proof of Theorem 2. Let Q be the aggregate of all points y of

Y with $\partial f^{-1}(y)=\phi$. For each y in Q select a point $x(y)$ from $f^{-1}(y)$. Let

$$P=\{x(y)\mid y\in Q\}\cup[\cup\{\partial f^{-1}(y)\mid y\notin Q\}].$$

Then P is closed in X. Since P has also a σ-locally finite net (cf. [7]) and $f|P$ is a closed mapping onto Y, it is sufficient to prove our Theorem in the case of $X=P$. Therefore, without loss of generality, we can assume that $\partial f^{-1}(y)=f^{-1}(y)$ for each $y\in Y$ with $\partial f^{-1}(y)\neq\phi$.

Let $\mathfrak{B}=\bigcup_{n=1}^{\infty}\mathfrak{B}_n$ be a σ-locally finite net for X satisfying all conditions in Lemma 1. In addition, since X is regular, each member of $\mathfrak{B}$ is a closed subset of X. Let

$$Y_n=\{y\mid f^{-1}(y)\cap B_\alpha=\phi \text{ for all but finite } B_\alpha \text{ in } \mathfrak{B}_n\}$$

for each n. Then each Y_n is open in Y. That is, for an arbitrary $y_0\in Y_n$ $V=Y-\cup\{f(B_\alpha)\mid B_\alpha\in\mathfrak{B}_n, y_0\notin f(B_\alpha)\}$ is an open subset of Y containing y_0, since $\mathfrak{B}_n$ is a locally finite closed covering of X and f is a closed mapping. Furthermore, if y is in V, we have that $f^{-1}(y)\cap B_\alpha\neq\phi$ implies $f^{-1}(y_0)\cap B_\alpha\neq\phi$ for $B_\alpha\in\mathfrak{B}_n$. This shows that V is contained in Y_n and, certainly, Y_n is open in Y.

Since X is perfectly normal (cf. [7]), Y is so, too. Hence, for each n we can put $Y'_n=Y-Y_n=\bigcap_{i=1}^{\infty}G_{ni}$ with decreasing sequence $\{G_{ni}\mid i=1,2,\cdots\}$ of open subsets of Y.

Now, let us put

$$\mathfrak{C}_{ni}=\{f(B_\alpha)-G_{ni}\mid B_\alpha\in\mathfrak{B}_n\} \quad\text{for}\quad i=1,2,\cdots;n=1,2,\cdots,$$

and show that each $\mathfrak{C}_{ni}$ is locally finite in Y. Let y be an arbitrary point of Y. If y is in G_{ni}, $\mathfrak{C}_{ni}$ is clearly locally finite at y. If y is not in G_{ni}, y must be in Y_n. By the construction of Y_n y is contained in only finite members of $f(\mathfrak{B}_n)=\{f(B_\alpha)\mid B_\alpha\in\mathfrak{B}_n\}$, in other words, the last collection is point-finite at y and, moreover, it is closure-preserving[6] in Y. Hence $f(\mathfrak{B}_n)$ is locally finite at y and, consequently, $\mathfrak{C}_{ni}$ is locally finite at y.

Finally, let us put

$$Y'=\{y\mid f^{-1}(y) \text{ is not countably compact}\}$$

and show that

$$\mathfrak{C}=\bigcup_{i,n=1}^{\infty}\mathfrak{C}_{ni}\cup\{\{y\}\mid y\in Y'\}$$

is a σ-locally finite net for Y. Since Y' is σ-discrete in Y by Theorem 1 and each $\mathfrak{C}_{ni}$ is locally finite in Y, it remains only to prove that $\mathfrak{C}$ is a net for Y. Let y be an arbitrary point of Y and U an arbitrary open subset of Y containing y. Since it is clear for y in Y', we can

6) A collection $\mathfrak{U}$ is said to be *closure-preserving* if for any $\mathfrak{B}\subset\mathfrak{U}$ $\overline{\cup\{V\mid V\in\mathfrak{B}\}}=\cap\{\overline{V}\mid V\in\mathfrak{B}\}$. For any locally finite collection $\mathfrak{U}$ in X $f(\mathfrak{U})$ is closure-preserving in Y by the closedness of f.

assume that $f^{-1}(y)$ is countably compact and, consequently, y is in Y_n for $n=1, 2, \cdots$. Since $\mathfrak{B}$ is a net for X, there exist an n and B_α in $\mathfrak{B}_n$ with $y \in f(B_\alpha) \subset U$. Since y is not in Y', we can choose k such that $y \notin G_{nk}$. Hence we have $y \in f(B_\alpha) - G_{nk} \subset U$ and $f(B_\alpha) - G_{nk} \in \mathfrak{C}_{nk}$. This shows that $\mathfrak{C}$ is a net for Y. This completes the proof.

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