104. A Remark on the Normal Expectations

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1. As concerns the channels in the mathematical theory of information, we have discussed under operator method and have introduced the notion of generalized channel in [3].

In this paper, we shall show a relation between certain generalized channels and normal expectations; that is, the conjugate mapping of a generalized channel having a special property is a normal expectation and the converse is true. Furthermore, by using this result, we shall study that for which type von Neumann algebra $\mathcal{A}$ on a Hilbert space $\mathfrak{H}$ there exists a faithful normal expectation of full operator algebra $L(\mathfrak{H})$ onto $\mathcal{A}$.

2. Consider a von Neumann algebra $\mathcal{A}$, denote the conjugate space as $\mathcal{A}^*$ and the subconjugate space of all ultraweakly continuous linear functionals on $\mathcal{A}$ as $\mathcal{A}_*$ basing on the definition of Dixmier [2].

Let $\mathcal{A}$ and $\mathcal{B}$ be two von Neumann algebras, then a positive linear mapping π of $\mathcal{A}_*$ into $\mathcal{B}_*$ is called a *generalized channel* if π preserves the norm of positive elements. Then the following proposition is obtained in [3].

Proposition 1. *A positive linear mapping π of $\mathcal{A}_*$ into $\mathcal{B}_*$ is a generalized channel if and only if the conjugate mapping π^* is a positive normal linear mapping of $\mathcal{B}$ into $\mathcal{A}$ preserving the identity.*

Let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{B}$ a von Neumann subalgebra of $\mathcal{A}$, then the positive linear mapping e of $\mathcal{A}$ onto $\mathcal{B}$ is called an expectation of $\mathcal{A}$ onto $\mathcal{B}$ if e satisfies the following equalities:

(1) $$I^e = I$$

(2) $$(BA)^e = BA^e \quad \text{for all } A \in \mathcal{A} \text{ and } B \in \mathcal{B}.$$

Define the operator L_A on $\mathcal{A}^*$ for each $A \in \mathcal{A}$ such that

(3) $$L_A f(X) = f(AX) \quad \text{for all } f \in \mathcal{A}^* \text{ and } X \in \mathcal{A},$$

then we have following theorem.

Theorem 2. *Let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{B}$ a von Neumann subalgebra of $\mathcal{A}$, then a mapping π of $\mathcal{B}_*$ to $\mathcal{A}_*$ is a generalized channel and*

(4) $$\pi L_B = L_B \pi \quad \textit{for any } B \in \mathcal{B}$$

if and only if the conjugate mapping π^* of $\mathcal{A}$ to $\mathcal{B}$ is a normal expectation.

Proof. If π is a generalized channel, then π^* is a positive normal linear mapping of $\mathcal{A}$ to $\mathcal{B}$ preserving the identity, by Proposition 1. By the equalities (3) and (4),

$$f(\pi^*(BX))=\pi f(BX)=L_B\pi f(X)=\pi L_B f(X)$$
$$=L_B f(\pi^*(X))=f(B\pi^*(X))$$
$$\text{for any } f\in\mathcal{B}_*,\ X\in\mathcal{A},\ \text{and } B\in\mathcal{B},$$

then we have

$$\pi^*(BX)=B\pi^*(X) \quad \text{for any } X\in\mathcal{A} \text{ and } B\in\mathcal{B}.$$

Therefore π^* is a normal expectation of $\mathcal{A}$ to $\mathcal{B}$.

Conversely, if π^* is a normal expectation of $\mathcal{A}$ to $\mathcal{B}$, by Proposition 1, π is a generalized channel, furthermore by the property (2) of expectation we have

$$\pi L_B=L_B\pi \quad \text{for any } B\in\mathcal{B}.$$

Theorem 3. *In above Theorem 2, the expectation π^* is faithful if and only if the generalized channel π satisfies the following condition:*

(5) *for a positive element X in $\mathcal{A}$, if $f(X)=0$ for any $f\in\pi(\mathcal{B}_*)$, then $X=0$.*

Proof. $\pi g(X)=g(\pi^*(X))=0$ for any g in $\mathcal{B}_*$ if and only if $\pi^*(X)=0$. Then a necessary and sufficient condition that π^* is faithful is the condition (5).

3. Let $\mathcal{A}$ be a von Neumann algebra and G a group of automorphisms of $\mathcal{A}$, from now on we shall call briefly *-automorphism automorphism. In this place, depending on the terminology of Kovács and Szücs [4], we shall call $\mathcal{A}$ *G-finite* if for any nonzero positive element T in $\mathcal{A}$ there exists a normal positive linear functional φ on $\mathcal{A}$ such that

$$\varphi(T)\neq 0 \text{ and } \varphi(\theta(S))=\varphi(S) \text{ for any } S \text{ in } \mathcal{A} \text{ and } \theta \text{ in } G.$$

Put $\mathcal{A}^G=\{T\in\mathcal{A};\ \theta(T)=T \text{ for any } \theta\in G\}$, then $\mathcal{A}^G$ is a von Neumann subalgebra of $\mathcal{A}$.

Suppose that $\mathcal{A}$ is G-finite, then there exists a mapping $T\to T'$ of $\mathcal{A}$ to $\mathcal{A}^G$ which satisfies the following properties (i)–(vi), and the converse is true [4];

(i) for every $T\in\mathcal{A}$ and every ultraweakly continuous linear functional σ on $\mathcal{A}$ which is invariant with respect to G, $\sigma(T)=\sigma(T')$,

(ii) $T\to T'$ is linear and strictly positive,

(iii) for $T\in\mathcal{A}$ and $S\in\mathcal{A}^G$, $(ST)'=ST'$ and $(TS)'=T'S$,

(iv) $T\to T'$ is ultraweakly and ultrastrongly continuous,

(v) for every $T\in\mathcal{A}^G$, $T=T'$,

(vi) $(\theta(T))'=T'$ for every $T\in\mathcal{A}$ and $\theta\in G$.

Let $\mathcal{A}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$ and $L(\mathfrak{H})$ the full operator algebra on $\mathfrak{H}$, then we can define an automorphism

θ_U, for every unitary operator U in $\mathcal{A}$, as following:

$$\theta_U(T)=UTU^* \quad \text{for any } T \in \boldsymbol{L}(\mathfrak{H}).$$

So, denote by $G(\mathcal{A}_U)$ a group of all automorphisms of $\boldsymbol{L}(\mathfrak{H})$ such as θ_U, then $\boldsymbol{L}(\mathfrak{H})$ is $G(\mathcal{A}_U)$-finite if and only if $\mathcal{A}$ is a product of finite type I factors [4, Proposition 5].

Now let us investigate a necessary and sufficient condition with respect to $G(\mathcal{A}_U)$ that for a finite von Neumann algebra $\mathcal{A}$ acting on a separable Hilbert space $\mathfrak{H}$, there exists a faithful normal expectation of $\boldsymbol{L}(\mathfrak{H})$ onto $\mathcal{A}$.

Lemma 1. *Let $\mathcal{A}$ be a I_n-factor on a countably infinite dimensional Hilbert space $\mathfrak{H}$, then there exists an isomorphism Φ of $\boldsymbol{L}(\mathfrak{H})$ onto $\mathcal{A}\otimes\boldsymbol{L}(\mathfrak{H})$ which transforms $\mathcal{A}$ onto $\mathcal{A}\otimes\boldsymbol{C}_{\mathfrak{H}}$.*

Proof. By the assumption that $\mathfrak{H}$ is countably infinite dimensional, there exists an isomorphism Ψ of $\boldsymbol{L}(\mathfrak{H})$ onto $\mathcal{A}\otimes\boldsymbol{L}(\mathfrak{H})$. Put $\mathcal{B}=\Psi(\mathcal{A})$ then $\mathcal{B}$ is a I_n-subfactor of $\mathcal{A}\otimes\boldsymbol{L}(\mathfrak{H})$. Since $\mathcal{A}\otimes\boldsymbol{C}_{\mathfrak{H}}$ is also a I_n-subfactor of $\mathcal{A}\otimes\boldsymbol{L}(\mathfrak{H})$, there is a unitary operator U in $\mathcal{A}\otimes\boldsymbol{L}(\mathfrak{H})$ such that $U\mathcal{B}U^{-1}=\mathcal{A}\otimes\boldsymbol{C}_{\mathfrak{H}}$ [6, Lemma 3.3]. For every $T\in\boldsymbol{L}(\mathfrak{H})$, put $\Phi(T)=U\Psi(T)U^{-1}$, then this mapping Φ is an isomorphism claimed in the lemma.

Lemma 2. *Let $\mathcal{A}$ be a I_n-factor on a separable Hilbert space $\mathfrak{H}$, then there exists a faithful normal expectation ε of $\boldsymbol{L}(\mathfrak{H})$ onto $\mathcal{A}$.*

Proof. If $\mathfrak{H}$ is finite dimentional, the lemma is clear by [9]. Suppose $\mathfrak{H}$ is countably infinite dimensional. It is sufficient to show that there exists a faithful normal expectation e of $\mathcal{A}\otimes\boldsymbol{L}(\mathfrak{H})$ onto $A\otimes\boldsymbol{C}_{\mathfrak{H}}$. In fact, suppose that there exists such a faithful normal expectation e. Put

$$T^{\varepsilon}=\Phi^{-1}((\Phi(T))^{e}) \quad \text{for all } T\in\boldsymbol{L}(\mathfrak{H}),$$

where Φ is the isomorphism of $\boldsymbol{L}(\mathfrak{H})$ onto $\mathcal{A}\otimes\boldsymbol{L}(\mathfrak{H})$ obtained in Lemma 1, then the mapping is a faithful normal expectation by the following equality:

$$(AT)^{\varepsilon}=\Phi^{-1}((\Phi(AT))^{e})=\Phi^{-1}(\Phi(A)(\Phi(T))^{e})=A\Phi^{-1}((\Phi(T))^{e})=AT^{\varepsilon}$$
$$\text{for any } A\in\mathcal{A} \text{ and } T\in\boldsymbol{L}(\mathfrak{H}).$$

Now, let us show that there exists a faithful normal expectation e of $\mathcal{A}\otimes\boldsymbol{L}(\mathfrak{H})$ onto $\mathcal{A}\otimes\boldsymbol{C}_{\mathfrak{H}}$. By the assumption that $\mathfrak{H}$ is separable, there exists a faithful normal state ψ_0 on $\boldsymbol{L}(\mathfrak{H})$. We define a mapping π of A_* to $(\mathcal{A}\otimes\boldsymbol{L}(\mathfrak{H}))_*$ by

$$\pi(\varphi)=\varphi\otimes\psi_0 \quad \text{for } \varphi\in\mathcal{A}_*,$$

where $\varphi\otimes\psi_0$ is the ultraweakly continuous linear functional on $\mathcal{A}\otimes\boldsymbol{L}(\mathfrak{H})$ such that

$$\varphi\otimes\psi_0(T_1\otimes T_2)=\varphi(T_1)\psi_0(T_2) \quad \text{for } T_1\in\mathcal{A} \text{ and } T_2\in\boldsymbol{L}(\mathfrak{H}).$$

Since $\mathcal{A}$ is a I_n-factor, there exists a faithful normal state φ_0 on $\mathcal{A}$. For this state φ_0, $\pi(\varphi_0)$ is faithful on $\mathcal{A}\otimes\boldsymbol{L}(\mathfrak{H})$, therefore π satisfies the condition (5). On the other hand, for every $A \in \mathcal{A}$ let us identity $A \otimes I$ and A, then the condition (4) is satisfied. Hence the conjugate mapping $e=\pi^*$ is a faithful normal expectation of $\mathcal{A} \otimes \boldsymbol{L}(\mathfrak{H})$ onto $\mathcal{A} \otimes \boldsymbol{C}_{\mathfrak{H}}$.

In Lemma 2, we can not exclude the assumption that $\mathfrak{H}$ is separable. That is, if there exists a faithful normal expectation of $\boldsymbol{L}(\mathfrak{H})$ onto a I_n-factor $\mathcal{A}$, then $\mathfrak{H}$ is separable. In fact, suppose that there exists a faithful normal expectation e of $\boldsymbol{L}(\mathfrak{H})$ onto a I_n-factor $\mathcal{A}$. Because $\mathcal{A}$ is a finite factor, there exists a faithful normal state φ on $\mathcal{A}$. For this state φ and the expectation e, we define a linear functional ψ as following:

$$\psi(T)=\varphi(T^e) \quad \text{for } T \in \boldsymbol{L}(\mathfrak{H}),$$

then ψ is a faithful normal state on $\boldsymbol{L}(\mathfrak{H})$, and so $\boldsymbol{L}(\mathfrak{H})$ is σ-finite [2], that is, $\mathfrak{H}$ is separable.

Theorem 4. *Let $\mathcal{A}$ be a finite von Neumann algebra acting on a separable Hilbert space $\mathfrak{H}$. Then the next three conditions are equivalent;*

(i) *there exists a faithful normal expectation e of $\boldsymbol{L}(\mathfrak{H})$ onto $\mathcal{A}$,*

(ii) *$\boldsymbol{L}(\mathfrak{H})$ is $G(\mathcal{A}_U)$-finite,*

(iii) *$\mathcal{A}$ is a product of finite type I factors.*

Proof. (ii) and (iii) are equivalent by [4, Proposition 5]. If there exists a faithful normal expectation e of $\boldsymbol{L}(\mathfrak{H})$ onto $\mathcal{A}$, then, for any finite normal trace φ on $\mathcal{A}$, we define a linear functional τ_φ on $\boldsymbol{L}(\mathfrak{H})$ by

$$\tau_\varphi(T)=\varphi(T^e) \quad \text{for any } T \in \boldsymbol{L}(\mathfrak{H}).$$

Then τ_φ is a normal positive linear functional on $\boldsymbol{L}(\mathfrak{H})$ such that

$$\tau_\varphi(UTU^*)=\tau_\varphi(T) \quad \text{for any } T \in \boldsymbol{L}(\mathfrak{H}) \text{ and any unitary } U \in \mathcal{A}.$$

By the assumptions that $\mathcal{A}$ is a finite von Neumann algebra and that e is faithful, for any nonzero positive element T in $\boldsymbol{L}(\mathfrak{H})$, there exists a finite normal trace φ_0 on $\mathcal{A}$ such that

$$\tau_{\varphi_0}(T)=\varphi_0(T^e) \neq 0.$$

Hence $\boldsymbol{L}(\mathfrak{H})$ is $G(\mathcal{A}_U)$-finite. Therefore we have that (i) implies (ii). Now let us show that (iii) implies (i). By the assumption, there exists a set $(E_\iota)_{\iota \in I}$ of projections in the center $\mathcal{Z}$ of $\mathcal{A}$ such that $\mathcal{A}=\prod_{\iota \in I} \mathcal{A}_{E_\iota}$ and that each $\mathcal{A}_{E_\iota}$ is a finite type I factor. Therefore by Lemma 2, for each $\iota \in I$, there exists a faithful normal expectation e on $\boldsymbol{L}(\mathfrak{H})_{E_\iota} = \boldsymbol{L}(E_\iota(\mathfrak{H}))$ onto $\mathcal{A}_{E_\iota}$. For any T in $\boldsymbol{L}(\mathfrak{H})$, define a mapping $T \rightarrow T^e$ of $\boldsymbol{L}(\mathfrak{H})$ to $\mathcal{A}$ by

$$T^e=\Pi(T_{E_\iota})^{e_\iota},$$

then the mapping $T \rightarrow T^e$ is clearly a faithful normal expectation.

For a condition of the existence of a normal expectation of $\boldsymbol{L}(\mathfrak{H})$ onto $\mathcal{A}$ which is not necessary faithful, it is discussed in [7] and [8].

As a result of them, the equivalence relation of (i) and (iii) in above Theorem 4 may have been known already.

In Theorem 4, (i) implies (ii) and (iii) without the separability of the Hilbert space $\mathfrak{H}$ which is clear by above proof. Then we have following, [1, Proposition 6.2.4]

Corollary 1. *Let $\mathcal{A}$ be a non-atomic abelian von Neumann algebra acting a Hilbert space $\mathfrak{H}$, then there does not exist a faithful normal expectation of $\boldsymbol{L}(\mathfrak{H})$ onto $\mathcal{A}$.*

Corollary 2. *Let $\mathcal{A}$ be a finite von Neumann algebra acting on a Hilbert space $\mathfrak{H}$ which is not necessary separable. If there exists a faithful normal expectation of $\boldsymbol{L}(\mathfrak{H})$ onto $\mathcal{A}$, there exists a faithful normal expectation of $\boldsymbol{L}(\mathfrak{H})$ onto $\mathcal{A}'$, too.*

Proof. By Theorem 4, $\boldsymbol{L}(\mathfrak{H})$ is $G(\mathcal{A}_U)$-finite. On the other hand $\boldsymbol{L}(\mathfrak{H})^{G(\mathcal{A}_U)}=\mathcal{A}'$. Therefore, by the theorem of Kovács-Szücs, there exists a faithful normal expectation of $\boldsymbol{L}(\mathfrak{H})$ onto $\mathcal{A}'$.

The converse of Corollary 2 is not true in generally. A trivial example is given by the von Neumann algebra $\boldsymbol{C}_{\mathfrak{H}}$ on an uncountably infinite dimensional Hilbert space $\mathfrak{H}$.

The next corollary is an immediate consequence of Theorems 2, 3, and 4.

Corollary 3. *Let $\mathcal{A}$ be a finite von Neumann algebra acting on a separable Hilbert space $\mathfrak{H}$. Then a necessary and sufficient condition that there exists a generalized channel of $\mathcal{A}_*$ onto $\boldsymbol{L}(\mathfrak{H})_*$ which satisfies the conditions* (4) *and* (5) *is that $\mathcal{A}$ is a product of finite type I factors.*

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