# 103. *Integration with Respect to the Generalized Measure.* IV

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(Comm. by Kinjirô KUNUGI, M. J. A., June 12, 1968)

In this part of the paper, we discuss an integral in a concrete form—an integral 'of a (measurable) function $f$ over a (measurable) set $X$' by a measure $\mu$.

1. **Definition of an integral system.** *Let $M$ be a non-empty set. Let $G$, $K$, and $J$ be topological additive groups*[1] *and assume that, for each $g \in G$ and $k \in K$, the product $g \cdot k$ of $g$ and $k$ is defined as an element of $J$ satisfying the conditions*:

1) $(g+g')\cdot k = g\cdot k + g'\cdot k$,

2) $g\cdot(k+k') = g\cdot k + g\cdot k'$,

*for each $g, g' \in G$, and $k, k' \in K$.*

Now let us denote by $\mathcal{F}$ the additive group of all $K$-valued functions defined on $M$ (the sum of two functions in $\mathcal{F}$ is defined in the usual way). We consider $\mathcal{F}$ as a topological group, in which the family of all sets of the form $\{f \mid f \in F,\ f(M) \subset P\}$, where $P$ is a neighbourhood of the unit element of $K$, constitutes a base of the system of neighbourhoods of the unit element of $\mathcal{F}$. This topology is characterized as the topology such that any sequence of elements of $\mathcal{F}$ converges in the space $\mathcal{F}$ if and only if the sequence uniformly converges as a functional sequence.

Then the map $\varphi$ of $K$ into $\mathcal{F}$ defined by $(\varphi(a))(x) = a$, for each $a \in K$ and $x \in M$, is an isomorphism of the topological group $K$ into $\mathcal{F}$ so that we may identify the topological group $K$, by the isomorphism $\varphi$, with the subgroup $\varphi(K)$ of $\mathcal{F}$.

Let $\mathcal{M}$ be the family of all subsets of $M$. Then $\mathcal{M}$ is a ring (in the algebraic sense) of which each element is an idempotent, when we define, for each $X$ and $Y$ in $\mathcal{M}$, $X+Y$ and $XY$ by $(X-Y)\cup(Y-X)$ and $X\cap Y$, respectively.

For each $X \in \mathcal{M}$ and $f \in \mathcal{F}$, denote $Xf$ the function in $\mathcal{F}$ such that

$$(Xf)(x) = \begin{cases} f(x) & \text{if } x \in X, \\ 0 & \text{if } x \in M-X. \end{cases}$$

Then each element $X$ of $\mathcal{M}$ is considered as a continuous homomorphism of $\mathcal{F}$ into itself satisfying the conditions:

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1) The topology of $G$ plays no role here.

1) $(X+Y)f=Xf+Yf$ if $XY=0$,

2) $(XY)f=X(Yf)$,

for each $X$, $Y \in \mathcal{M}$, and $f \in \mathcal{F}$.

With the situation above, the system $(M, G, K, J)$ is called an *integral system* and $M$, $G$, $K$, $J$, $\mathcal{F}$, and $\mathcal{M}$ are called the *base space*, the *first group*, the *second group*, the *third group*, the *total functional group*, and the *total ring*, respectively, of this integral system.

**2. Definitions of an integral structure and an integral.** Let $\mathcal{S}$ be a ring (in the algebraic sense) of which each element is an idempotent and $G$ a topological additive group. Then the pair $(\mathcal{S}, G)$ is called a *measure system*. For a measure system $(\mathcal{S}, G)$, a map $\mu$ of $\mathcal{S}$ into $G$ is called a *G-valued pre-measure* on $\mathcal{S}$ (or a pre-measure *with respect to* $(\mathcal{S}, G)$) if it satisfies the condition:

$\mu(X+Y)=\mu(X)+\mu(Y)$ for each $X$, $Y$ in $\mathcal{S}$ such that $XY=0$.

The set $\mathcal{P}$ of all pre-measures with respect to a measure system $(\mathcal{S}, G)$ forms an additive group when we define the sum $\mu+\nu$ of two elements $\mu$, $\nu$ of $\mathcal{P}$ by $(\mu+\nu)(X)=\mu(X)+\nu(X)$, for each $X \in \mathcal{S}$. This additive group $\mathcal{P}$ is called the *total pre-measure group* of the measure system $(\mathcal{S}, G)$.

Suppose $\Lambda$ is an integral system. A subgroup $\mathcal{G}$ of the total functional group of $\Lambda$ is called a *functional group* of $\Lambda$ if it contains $K$, and a subring $\mathcal{S}$ of the total ring of $\Lambda$ is called a *measurable ring* of $\Lambda$. For a measurable ring $\mathcal{S}$ of $\Lambda$, the pair $(\mathcal{S}, G)$, $G$ being the first group of $\Lambda$, forms a measure system *determined by* $\Lambda$ and $\mathcal{S}$. A subset $Q$ of the total pre-measure group of $(\mathcal{S}, G)$ is called a *pre-measure space* of $\Lambda$ *relative to* $\mathcal{S}$.

Let $\Lambda=(M, G, K, J)$ be an integral system. Then, for a measurable ring $\mathcal{S}$ of $\Lambda$, for an $\mathcal{S}$-invariant functional group $\mathcal{G}$ of $\Lambda$, and for a pre-measure space $Q$ of $\Lambda$ relative to $\mathcal{S}$, the system $(M, G, K, J; \mathcal{S}, \mathcal{G}, Q)$ $=(\Lambda; \mathcal{S}, \mathcal{G}, Q)$ is called an *integral structure* and $M$, $G$, $K$, $J$, $\mathcal{S}$, $\mathcal{G}$, $Q$, and $\Lambda$ are called the *base space*, the *first group*, the *second group*, the *third group*, the *measurable ring*, the *functional group*, the *pre-measure space*, and the *integral system*, respectively, of the integral structure.

If $\Gamma$ is an integral structure, the system $(\mathcal{S}, \mathcal{G}, J)$ of the measurable ring $\mathcal{S}$, the functional group $\mathcal{G}$, and the third group $J$ of $\Gamma$ forms an abstract integral structure *derived from* the integral structure $\Gamma$.

Let $\Gamma=(M, G, K, J; \mathcal{S}, \mathcal{G}, Q)$ be an integral structure and $\sigma$ a map of $\mathcal{S}\times\mathcal{G}\times Q$ into $J$. Suppose that, for any fixed $\mu \in Q$, the map $\mathcal{J}_\mu$ $=\mathcal{J}_\mu(X, g)=\sigma(X, g, \mu)$ of $\mathcal{S}\times\mathcal{G}$ into $J$ is an abstract integral with respect to the derived abstract integral structure $(\mathcal{S}, \mathcal{G}, J)$ from $\Gamma$; or that the map $\sigma$ of $\mathcal{S}\times\mathcal{G}\times Q$ into $J$ satisfies the conditions:

( $*$ ) The map $\sigma=\sigma(X, g, \mu)$ is a continuous homomorphism of $\mathcal{G}$ into $J$ with respect to $g \in \mathcal{G}$ for any fixed $X \in \mathcal{S}$ and $\mu \in Q$.

( $**$ ) $\sigma(XY, g, \mu)=\sigma(X, Yg, \mu)$ for each $X, Y \in \mathcal{S}$, $g \in \mathcal{G}$, and $\mu \in Q$.

Then the map $\sigma$ is called an *integral* with respect to $\Gamma$ if it satisfies

($***$) $\sigma(X, a, \mu)=\mu(X)\cdot a$ for each $X \in \mathcal{S}$, $a \in K$, and $\mu \in Q$.

If $\sigma$ is an integral, the abstract integral $\mathcal{J}_\mu$ defined above is called the *derived* abstract integral *from $\sigma$ relative to $\mu$*.

**3. Propositions and a theorem.** **Proposition 1.** *Let $\Gamma=(\Lambda; \mathcal{S}, \mathcal{G}, Q)$ and $\Gamma'=(\Lambda; \mathcal{S}, \mathcal{G}', Q')$ be integral structures such that $\mathcal{G}'\subset\mathcal{G}$ and $Q'\subset Q$. Then the restriction of any integral with respect to $\Gamma$ on $\mathcal{S}\times\mathcal{G}'\times Q'$ is an integral with respect to $\Gamma'$.*

**Proof.** This follows from the lemma below, which is easily verified.

**Lemma 1.** *Let $(\mathcal{S}, \mathcal{F}, J)$ be an abstract integral structure, $\mathcal{S}'$ a subring of $\mathcal{S}$ and $\mathcal{F}'$ an $\mathcal{S}'$-invariant subgroup of $\mathcal{F}$. Then $(\mathcal{S}', \mathcal{F}', J)$ is an abstract integral structure and, for any abstract integral $\mathcal{J}$ with respect to $(\mathcal{S}, \mathcal{F}, J)$, the restriction of $\mathcal{J}$ on $\mathcal{S}'\times\mathcal{F}'$ is an abstract integral with respect to $(\mathcal{S}', \mathcal{F}', J)$.*

Let $\Lambda=(M, G, K, J)$ be an integral system and $\mathcal{S}$ a measurable ring of $\Lambda$. Then, for the total functional group $\mathcal{F}$ of $\Lambda$, the system $(\mathcal{S}, \mathcal{F}, J)$ forms an abstract integral structure *determined by $\Lambda$* and $\mathcal{S}$. The integral closure [1] $\mathcal{G}$ of $K$ in $\mathcal{F}$ with respect to $(\mathcal{S}, \mathcal{F}, J)$ is an $\mathcal{S}$-invariant functional group of $\Lambda$, which is called the *fundamental functional group* of $\Lambda$ *determined by* $\mathcal{S}$.

**Proposition 2.** *Let $\Gamma=(\Lambda; \mathcal{S}, \mathcal{G}, Q)$ be an integral structure. Suppose that $\mathcal{G}$ is a subgroup of the fundamental functional group of $\Lambda$ determined by $\mathcal{S}$ and that the third group of $\Lambda$ is a Hausdorff space. Then the integral with respect to $\Gamma$ is unique if it exists.*

**Proof.** Let $\mu$ be an element of $Q$. It suffices to show that the derived abstract integral $\mathcal{J}$ from an integral with respect to $\Gamma$ relative to $\mu$ is uniquely determined. Put $\Lambda=(M, G, K, J)$ and denote by $\mathcal{F}$ the total functional group of $\Lambda$. Then $(\mathcal{S}, \mathcal{F}, J)$ is an abstract integral structure. Let us define a map $i$ of $\mathcal{S}\times K$ into $J$ by $i(X, a)=\mu(X)\cdot a$, for each $X \in \mathcal{S}$ and $a \in K$. Then, denoting by $\mathcal{G}_0$ the subgroup of $\mathcal{F}$ generated by $\mathcal{S}K$ and by $\mathcal{G}_1$ the $\mathcal{F}$-completion of $\mathcal{G}_0$, we see that the conditions in Assumption 1 in [3] is satisfied if we read $\mathcal{G}_1$ for $\mathcal{G}$. When we denote by $\mathcal{G}_2$ the $\mathcal{F}$-completion of $\mathcal{G}$, Proposition 3.16 in [1] implies that $\mathcal{J}$ is uniquely extended to an abstract integral $\mathcal{J}_2$ with respect to $(\mathcal{S}, \mathcal{G}_2, J)$. It holds that $K\subset\mathcal{G}_1\subset\mathcal{G}_2$ and the restriction $\mathcal{J}_1$ of $\mathcal{J}_2$ on $\mathcal{S}\times\mathcal{G}_1$ is an abstract integral with respect to $(\mathcal{S}, \mathcal{G}_1, J)$ which is an extension of $i$. Hence it follows from Propoition 3 in [3] that $\mathcal{J}_1$

is unique. Since the integral closure $\tilde{\mathcal{G}}_1$ of $\mathcal{G}_1$ coincides with the fundamental functional group of $\Lambda$, we have $\mathcal{G}\subset\tilde{\mathcal{G}}_1$ and therefore we have $\mathcal{G}_2\subset\tilde{\mathcal{G}}_1$. Hence the fact that $\mathcal{J}_2$ is an extension of $\mathcal{J}_1$ and the following lemma imply that $\mathcal{J}_2$ is uniquely determined. This implies that $\mathcal{J}$ is unique.

**Lemma 2.** *Let $(\mathcal{S}, \mathcal{F}, J)$ be an abstract integral structure and let $\mathcal{G}_1$ and $\mathcal{G}$ be $\mathcal{S}$-invariant subgroups of $\mathcal{F}$ such that $\mathcal{G}_1\subset\mathcal{G}\subset\tilde{\mathcal{G}}_1$, where $\tilde{\mathcal{G}}_1$ is the integral closure of $\mathcal{G}_1$ in $\mathcal{F}$. Assume that $J$ is a Hausdorff space. Then, if an abstract integral $\mathcal{J}_1$ with respect to $(\mathcal{S}, \mathcal{G}_1, J)$ is extended to an abstract integral $\mathcal{J}$ with respect to $(\mathcal{S}, \mathcal{G}, J)$, such an extension $\mathcal{J}$ is uniquely determined.*

**Proof.** It follows from Corollary to Proposition 3.17 in [1] that $\tilde{\mathcal{G}}_1$ is the $\mathcal{F}$-completion of the closure $\bar{\mathcal{G}}_1$ of $\mathcal{G}_1$ in $\mathcal{F}$. Hence, the formula $\mathcal{G}\subset\tilde{\mathcal{G}}_1$ implies that the perfection $\mathcal{G}'$ of $\mathcal{G}$ is contained in $\bar{\mathcal{G}}_1$ (Proposition 3.9 in [1]). Thus the facts that the map $\mathcal{J}=\mathcal{J}(X, g)$, for fixed $X\in\mathcal{S}$, of $\mathcal{G}$ into $J$ is continuous on $\mathcal{G}$ which contains $\mathcal{G}_1$ and $\mathcal{G}'$ and that $J$ is a Hausdorff space imply that $\mathcal{J}$ is uniquely determined by $\mathcal{J}_1$ on $\mathcal{S}\times\mathcal{G}'$. The uniqueness of $\mathcal{J}$ on $\mathcal{S}\times\mathcal{G}$ follows from this and the lemma is proved.

Let $\Lambda=(M, G, K, J)$ be an integral system and $\mathcal{S}$ a measurable ring of $\Lambda$. Denote by $\mathcal{F}$ the total functional group of $\Lambda$.

For a pre-measure $\mu$ with respect to the measure system $(\mathcal{S}, G)$ determined by $\Lambda$ and $\mathcal{S}$, we say $\mu$ is *integrable* if the following condition is satisfied: for any $X$ in $\mathcal{S}$ and for any neighbourhood $V$ of the unit element of $J$, there exists a neighbourhood $P$ of the unit element of $\mathcal{F}$ such that

$$a_i\in P\cap K,\ X_i\in\mathcal{S},\ i=1,2,\cdots,n, \quad \text{and} \quad X_jX_k=0\ (j\neq k)$$
$$\text{imply}\quad \sum_{i=1}^{n}\mu(XX_i)\cdot a_i\in V.$$

The set of all integrable $G$-valued pre-measures on $\mathcal{S}$ forms a subgroup of the total pre-measure group of $(\mathcal{S}, G)$, which is called the *fundamental pre-measure group* of $\Lambda$ *determined by* $\mathcal{S}$.

Let $\mathcal{G}$ and $Q$ be the fundamental functional group and the fundamental pre-measure group, respectively, of $\Lambda$ determined by $\mathcal{S}$. Then the system $\Gamma=(\Lambda; \mathcal{S}, \mathcal{G}, Q)$ is an integral structure. This integral structure $\Gamma$ is called the *fundamental integral structure determined by* $\Lambda$ and $\mathcal{S}$.

Now we can state our main theorem:

**Theorem 1.** *Let $\Gamma$ be a fundamental integral structure and assume that the third group of $\Gamma$ is a Hausdorff, complete group. Then there exists a unique integral with respect to $\Gamma$.*

**Proof.** Put $\Gamma=(\Lambda; \mathcal{S}, \mathcal{G}, Q)$ and $\Lambda=(M, G, K, J)$. Denote by $\mathcal{F}$

the total functional group of $\Lambda$. Then $(\mathcal{S}, \mathcal{F}, J)$ is an abstract integral structure and $K$ is a subgroup of $\mathcal{F}$. For each $\mu \in Q$, let us define a map $i_\mu$ of $\mathcal{S} \times K$ into $J$ by

$$i_\mu(X, a) = \mu(X) \cdot a \quad \text{for each } X \in \mathcal{S} \text{ and } a \in K,$$

and denote by $\mathcal{G}_0$ the subgroup of $\mathcal{F}$ generated by $\mathcal{S}K$ and by $\mathcal{G}_1$ the $\mathcal{F}$-completion of $\mathcal{G}_0$. Then it is easily verified that the conditions in Assumptions 1, 2, 3, and 4 in [3] are satisfied, when we read $i_\mu$ for $i$ and $\mathcal{G}_1$ for $\mathcal{G}$. Hence Theorem 1 in [3] implies that the map $i_\mu$ is uniquely extended to an abstract integral $\mathcal{J}_\mu$ with respect to $(\mathcal{S}, \mathcal{G}_1, J)$.

Since $\mathcal{G}$ is the integral closure of $K$ and since $K \subset \mathcal{G}_1 \subset \mathcal{G}$, it follows that $\mathcal{G}$ is the integral closure of $\mathcal{G}_1$. Thus Theorem 1 in [1] implies that $i_\mu$ is uniquely extended to an abstract integral $\tilde{\mathcal{J}}_\mu$ with respect to $(\mathcal{S}, \mathcal{G}, J)$. Defining a map $\sigma$ of $\mathcal{S} \times \mathcal{G} \times Q$ into $J$ by $\sigma(X, g, \mu) = \tilde{\mathcal{J}}_\mu(X, g)$, for each $X \in \mathcal{S}$, $g \in \mathcal{G}$, and $\mu \in Q$, we have an integral with respect to $\Gamma$. The uniqueness of such an integral follows from Proposition 2 and thus the theorem is proved.

**Corollary.** *Let $\Gamma = (\Lambda; \mathcal{S}, \mathcal{G}, Q)$ be an integral structure with a Hausdorff, complete third group, and suppose that $\mathcal{G}$ and $Q$ are contained in the fundamental functional group and in the fundamental pre-measure group, respectively, of $\Lambda$ determined by $\mathcal{S}$. Then there exists a unique integral with respect to $\Gamma$ and this integral is the restriction of the integral in Theorem* 1.

**Proof.** This follows immediately from Propositions 1 and 2.

**Remark.** The following is easily verified: if there exists an integral with respect to an integral structure $(\Lambda; \mathcal{S}, \mathcal{G}, Q)$, the pre-measure space $Q$ is necessarily contained in the fundamental pre-measure group of $\Lambda$ determined by $\mathcal{S}$.

## References

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