99. Generalizations of the Alaoglu Theorem with Applications to Approximation Theory. II

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(Comm. by Kinjirô KUNUGI, M. J. A., June 12, 1968)

We will use the same notations as those in Part I.

7. **Theorem.** *Let $E_1, \cdots, E_n, E, F, X$, and Q have the same meaning as in Theorem* 3. Let

$$Y=\{[\lambda_1 x_1, \cdots, \lambda_n x_n]: |\lambda_i|\leq 1, i=1, \cdots, n, [x_1, \cdots, x_n]\in X\}$$
$$Z_i=\{[y_1, \cdots, y_{i-1}, \lambda y_i+(1-\lambda)y'_i, \cdots, y_n]: 0\leq\lambda\leq 1,$$
$$[y_1, \cdots, y_{i-1}, y_i, y_{i+1}, \cdots, y_n]\in Y$$
$$[y_1, \cdots, y_{i-1}, y_i, y_{i+1}, \cdots, y_n]\in Y\}$$
$$[X]=\cup\{Z_i: i=1, \cdots, n\}.$$

Suppose that 0 lies in the interior of the closure of $[X]$:

(C) $$0\in \text{int}\ \overline{[X]}.$$

Then, for each $k\geq 0$, the set $\{A\in b(E, F): ||A-Q||_X\leq k\}=S_k$ is σ-compact and σ-closed.

Proof. The proof is very similar to that of Theorem 3. Thus, for all $x\in X$ and all $A\in S_k$,

$$||Ax||\leq||Q||_X+k.$$

This inequality is valid if x ranges over the sets X, Y, $Z_i\,(i=1, \cdots, n)$ $[X]$ and, by continuity, $\overline{[X]}$. By Condition (C) the set contains an open sphere with radius $2r>0$. Then, for each $y\in E$,

$$||Ay||=||A\left(\frac{||y||}{r}\frac{r}{||y||}\right)||\leq\frac{||y||^n}{r^n}(||Q||_X+k)\equiv k'||y||^n.$$

It follows that

$$S_k\subseteq\prod_y\{f\in F: ||f||\leq k'||y||^n\}$$

where the product on the right is compact in the product topology. By arguments similar to those in the proof of Theorem 3, we can easily show that any net in S_k has a subnet which converges in the σ-topology to an element of S_k, thus proving that S_k is σ-compact. Since a net in $b(E, F)$ converges to at most one limit in the σ-topology, the space is Hausdorff. Consequently, the σ-compactness implies the σ-closedness.

8. **Corollary.** *Let E_1 and E_2 be normed linear spaces. Let X_1 be a subset of E_1 such that* 0 *lies in the interior of the closed convex balanced extension of X_1. Let Q be a set-valued bounded map of X_1 into the dual space E_2^* of E_2. Then, Q has a best approximation in any τ-closed subset of $B(E_1, E_2^*)$.*

Proof. Take, in the last theorem, $n=2$, $F=$the scalar field, and $X=\{[x_1, x_2]: x_1 \in X_1, x_2 \in$ the unit sphere of $E_2\}$. Using the isometric isomorphism between $B(E_1, E_2^*)$ and $b(E, +E_2, F)$ significant by the correspondence $A \in B(E_1, E_2^*) \leftrightarrow <*, A^*>$ [3, p. 102], we easily prove the present corollary.

9. Remark. The theorem given in [1, p. 97] is identical with the last corollary except that the map Q is single-valued. As before, Theorem 7 includes the Alaoglu theorem.

10. Corollary. Let E be a normed linear space and M any finite-dimensional subspace of E. Then, among all bounded linear projections of E onto M, there exists one with minimum norm. Also, there exists one which best approximates the Tchebycheff map T_M, the map which assigns to each element x of E the set of all best approximations in M to x.

Proof. Take, in the last corollary, $E_2=M^*(M^{**}=M)$, $X_1=$the unit sphere of E, and $Q=0$ or $Q=T_M$. Let P be the set of all bounded linear projections of E onto M. We must show that P is τ-closed in $B(E, M)$. To this end, let A_α be a net in P and let $A_\alpha \to A \in B(\mathrm{E}, M)$ in the τ-topology. This last condition implies that, for all $m \in M$ and all $m^* \in M^*$,

$$<Am, m^*>=\lim <A_\alpha m, m^*>=<m, m^*>.$$

Hence, $Am=m$ for all m in M and A is a projection onto M.

11. Remark. Let E be the direct sum of the normed linear spaces $E_1, \cdots, E_n$ and F a finite-dimensional normed linear space. Let X be a subset of E satisfying Condition (C) in Theorem 7. Then, according to the Theorems 3 and 7, any bounded set-valued map Q of X into F has best approximations in $u(E, F)$ and in $b(E, F)$, where $E=E_1\oplus \cdots \oplus E_n$. Best approximations in the space $u(E, F)$ are generally better than those in $b(E, F)$, since $b(E, F) \subseteq u(E, F)$. We will show next, by an example, that best approximations in $u(E, F)$ are sometimes strictly better than those in $b(E, F)$.

12. Example. Let E_1 be an infinite-dimensional separable Banach space, E_2 a normed linear space and $F(\neq\{0\})$ a finite-dimensional normed linear space. Let X_1 be a countable dense subset of the unit sphere of E_1. Let Y be the subset of $E_1 \oplus E_2$ defined by

$$Y=\{(x_1, x_2): x_1 \in X_1, \|x_2\| \leq 1\}.$$

Since X_1 is countable, it cannot span the whole space E_1, for otherwise we would have a contradition of the Baire theorem. Therefore, there exists a point y_1 not in the linear span of X_1. Let $y_2 \in E_2$ be arbitrary. Let

$$X=Y \cup \{(y_1, y_2)\}.$$

This set X satisfies Condition (C) of Theorem 7. Let Q be a map of Y

into F such that $Q=0$ on Y and $Q(y_1, y_2)\neq 0$. Then, Q is bounded. Using a Hamal base in $E_1\oplus E_2$, we see that Q can be bilinearly extended to the entire space $E_1\oplus E_2$. Hence, best approximations in $u(E_1\oplus E_2, F)$ to Q are exact. On the other hand, any best approximation in $b(E_1\oplus E_2, F)$ gives a positive error at the point (y_1, y_2) since the set X is dense in the unit sphere of $E_1\oplus E_2$ and $Q=0$ on X.

13. **Remark.** In Theorem 7, Condition (C) is sufficient to ensure a best approximation in $b(E, F)$. Is it also necessary? Unfortunately, we cannot answer this question at this moment. However, let us remark some related results in this regard.

Let E be a Banach space and X a subset of it. Let X^+ be the closed convex balanced extension of X. Consider the following condition on X:

(C)′ 0 is in the interior or X^+ relative to the linear span of X^+.

Cheney and Goldstein [1, p. 93] show that Condition (C)′ is sufficient in order that every bounded functional on X has a best approximation in the dual space E^*. The Hahn-Banach extension theorem is a key for this sufficiency. Kripke and Rockafellar [4, p. 1037] prove that, in case the set X is bounded, Condition (C)′ is also necessary. They prove this by showing the existence of a linear functional x^* on the linear span of X^+ such that x^* cannot be obtained as the restriction of any member of E^* to the linear span of X^+ and such that x^* can be approximated uniformly on X by elements of E^* with arbitrarily small positive error. The existence of such a linear functional x^* is based on the fact that, if Condition (C)′ fails, then the closure of the linear span of X^+ is strictly larger than the linear span itself. (The completeness of E and the boundedness of X are used to prove this last fact.)

Let F be a nonzero normed linear space. Suppose that the Condition (C)′ fails. Let x^* be the same functional as in the above. Take any non-zero element f in F. By considering the map Q of X into F defined by $Qx=<x, x^*>f$, the theorem in the next section is clear.

14. **Theorem.** Let E be a Banach space, X a nonempty bounded subset of E and F a nonzero normed linear space. If Condition (C)′ fails, then there exists a map of X into F which does not have a best approximation in $B(E, F)$.

References

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