93. *Fourier Series of Functions of Bounded Variation*

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Let f be an integrable function with period 2π and let

$$(1)\qquad f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx).$$

The following theorems are well known ([1], pp. 48, 57–58; [2] pp. 71–72, 114–116):

Theorem 1. *If f is of bounded variation, then*

$$(2)\qquad |a_n|\leqq V/\pi n,\quad |b_n|\leqq V/\pi n\quad \text{for all } n>1,$$

where V is the total variation of f over $(0, 2\pi)$.

Theorem 2. *If f is of bounded variation, then the Fourier series* (1) *converges to* $\frac{1}{2}(f(x+0)+f(x-0))$ *for every x.*

Recently, M. Taibleson [3] has given an elementary proof of Theorem 1, except the constant V/π in (2), which is the best possible. We shall give elementary proofs of Theorems 1 and 2.

Proof of Theorem 1.

$$(3)\quad \pi a_n=\int_0^{2\pi}f(x)\cos nx\,dx=\int_{-\pi/2n}^{2\pi-\pi/2n}=\sum_{k=0}^{2n-1}\int_{(k-1/2)\pi/n}^{(k+1/2)\pi/n}$$

$$=\sum_{k=0}^{2n-1}(-1)^k\int_{-\pi/2n}^{\pi/2n}f(x+k\pi/n)\cos nx\,dx$$

$$=\int_{-\pi/2n}^{\pi/2n}\left[\sum_{j=0}^{n-1}(f(x+2j\pi/n)-f(x+(2j+1)\pi/n))\right]\cos nx\,dx$$

$$=-\int_{-\pi/2n}^{\pi/2n}\left[\sum_{j=0}^{n-1}(f(x+(2j+1)\pi/n)-f(x+(2j+2)\pi/n))\right]\cos nx\,dx$$

and then

$$2\pi|a_n|\leqq\int_{-\pi/2n}^{\pi/2n}\left[\sum_{k=0}^{2n-1}|f(x+k\pi/n)-f(x+(k+1)\pi/n)|\right]\cos nx\,dx$$

$$\leqq V\int_{-\pi/2n}^{\pi/2n}\cos nx\,dx=2V/n.$$

Thus we get $|a_n|\leqq V/\pi n$. Similarly for b_n.

Proof of Theorem 2. We can suppose $f(x)=\frac{1}{2}[f(x+0)+f(x-0)]$ for all x. We put $f_x(t)=f(x+t)+f(x-t)-2f(x)$, then $f_x(t)$ is continuous at $t=0$. We denote by M the upper bound of $|f_x(t)|$ and by $V(a, b)$ the total variation of f_x on the interval (a, b), then we can easily see that

$$(4) \qquad \lim_{\varepsilon\to 0} V(0,\varepsilon)=0.$$

Further we denote by $s_n(x)$ the nth partial sum of the Fourier series (1), then,[1] by Theorem 1,

$$\begin{aligned} s_n(x)-f(x)&=\frac{1}{\pi}\int_0^\pi \frac{\sin nt}{t}f_x(t)dt+o(1)=\frac{1}{\pi}\sum_{k=0}^{n-1}\int_{k\pi/n}^{(k+1)\pi/n}+o(1)\\ &=\frac{1}{\pi}\int_0^{\pi/n}\left[\sum_{j=0}^{(n-1)/2}\left(\frac{f_x(t+2j\pi/n)}{t+2j\pi/n}-\frac{f_x(t+(2j+1)\pi/n)}{t+(2j+1)\pi/n}\right)\right]\sin nt\,dt+o(1)\\ &=\frac{1}{\pi}\int_0^{\pi/n}\left[\sum_{j=0}^{\varepsilon n}+\sum_{j=\varepsilon n+1}^{(n-1)/2}\right]\sin nt\,dt+o(1)=(I+J)+o(1), \end{aligned}$$

where

$$\begin{aligned} (5)\qquad |I|&\leqq\frac{1}{\pi}\int_0^{\pi/n}\left[\sum_{j=0}^{\varepsilon n}\frac{|f_x(t+2j\pi/n)-f_x(t+(2j+1)\pi/n)|}{t+2j\pi/n}\right]\sin nt\,dt\\ &\quad+\frac{1}{n}\int_0^{\pi/n}\left[\sum_{j=0}^{\varepsilon n}\frac{|f_x(t+(2j+1)\pi/n)|}{(t+2j\pi/n)(t+(2j+1)\pi/n)}\right]\sin nt\,dt\\ &\leqq 2V(0,3\varepsilon\pi)+2\sup_{0\leqq t\leqq 3\varepsilon\pi}|f_x(t)| \end{aligned}$$

for $n>1/\varepsilon$, and similarly

$$|J|\leqq\frac{V(2\varepsilon\pi,\pi)}{2\varepsilon\pi n}+\frac{M}{2\varepsilon\pi n}.$$

Therefore

$$\limsup_{n\to\infty}|s_n(x)-f(x)|\leqq\limsup_{n\to\infty}|I|,$$

where the right side tends to zero as $\varepsilon\to 0$, by (4) and (5).

Remarks. If we denote by $\omega(h)$ the modulus of continuity of f, then the formula (3) gives

$$(6)\qquad |a_n|\leqq\frac{2}{\pi}\,\omega\left(\frac{\pi}{n}\right),\qquad |b_n|\leqq\frac{2}{\pi}\,\omega\left(\frac{\pi}{n}\right)$$

(cf. [1], p. 45). In particular, if $f\in\mathrm{Lip}\,\alpha$ $(0<\alpha\leqq 1)$, that is

$$|f(t+h)-f(t)|\leqq Mh^\alpha \qquad \text{for all } t \text{ and } h>0,$$

then (6) becomes

$$|a_n|\leqq 2M/\pi^{1-\alpha}n^\alpha,\qquad |b_n|\leqq 2M/\pi^{1-\alpha}n^\alpha.$$

If we write $\Delta_h^2 f(x)=f(x+h)-2f(x)+f(x-h)$, then (3) becomes

$$(7)\qquad 2\pi a_n=\int_{-\pi/2n}^{\pi/2n}\left(\sum_{j=0}^{n-1}\Delta_{\pi/n}^2 f(x+2j\pi/n)\right)\cos nx\,dx.$$

We have defined the second modulus of continuity $\omega_2(h)$ by

$$\sup_{0\leqq x\leqq 2\pi}|f(x+h)-2f(x)+f(x-h)|,$$

then (7) gives

$$|a_n|\leqq\frac{1}{\pi}\,\omega_2\left(\frac{\pi}{n}\right),\qquad |b_n|\leqq\frac{1}{\pi}\,\omega_2\left(\frac{\pi}{n}\right).$$

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1) $\sum_{n=A}^{B}$ denotes the summation for $a\leqq n\leqq b$, where a and b are not necessarily integers.

References

[1] A. Zygmund: Trigonometric Series. I. Cambridge Univ. Press (1959).
[2] N. Bari: A Treatise on Trigonometric Series. I. Pergamon Press (1964).
[3] M. Taibleson: Fourier coefficients of functions of bounded variation. Proc. Amer. Math. Soc., **18**, 766 (1967).