181. On Nuclear Spaces with Fundamental System of Bounded Sets. II

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(Comm. by Kinjirô KUNUGI, M. J. A., Oct. 12, 1968)

A locally convex vector space with a countable fundamental system of bounded sets has already been developed in several bibliographies. Barrelled spaces and quasi-barrelled spaces with a countable fundamental system of compact sets has been studied by J. Dieudonné [2] and by M. Mahowald and G. Gould [7] respectively.

We considered, the open mapring and closed graph theorems on a nuclear dualmetric space in the previous paper [4].

Let E be a normed space then E is a nuclear space if and only if it is finite dimentional. It is also known that a normed space can only be a Montel (i.e., barrelled and perfect) space if it is finite dimensional. In this paper, we prove a nuclear dualmetric space which is quasi-complete is Montel space, and using this result, we consider analogous theorem to M. Mahowald and G. Gould [7], in nuclear space.

For nuclear spaces and its related notion, see A. Pietsch [8] and S. Funakosi [4]. Most of the definitions and notations of the locally convex vector spaces are taken from N. Bourbaki [1] and T. Husain [5].

Definition. *Let E be a locally convex space and E' its dual.*

(1) *If only all countable strong bounded subset of E' are equicontinuous, then E is called the σ-quasi-barrelled.*

(2) *Let E be a σ-quasi-barrelled space, if there exists a countable fundamental system of bounded subset in E, then E is called the dualmetric space.*

The following Lemma is well known.

Lemma 1. *A metric or dualmetric locally convex vector space E is nuclear if and only if its dualnuclear.*

The proof is given in A. Pietsch [8].

Proposition 1. *Each nuclear dualmetric space E is a quasi-barrelled.*

Proof. By Lemma 1, the strong dual E'^{β} is nuclear, so an arbitrary bounded subset of E'^{β} is separable (see, the proof of Theorem 4, (a) in S. Funakosi [4]). Denote by B strong bounded subset of E', then $B \subseteq \overline{\{a_n\,;\,a_n \in B\}}$. On the other hand, since E is dualmetric it is a

σ-quasi-barrelled, so there exist a neighborhood U such that $a_n \in U^0$ for every n, where U^0 is a polar of U. Therefore an arbitrary strong bounded subset of E' is a equicontinuous. Hence E is a quasi-barrelled space.

Corollary 1. *A nuclear dualmetric space is a Mackey space.*

Proof. By proposition, nuclear dualmetric space is a quasi-barrelled space. Moreover, quasi-barrelled space is a Mackey space (cf. [5], p. 31). Therefore nuclear dualmetric space is a Mackey space.

We remark, clearly a dualmetric space is a (DF)-space in G. Köthe [6] or A. Grothendieck [3]. Therefore, we have the following Lemma by G. Köthe [9, p. 405 (3), a)].

Lemma 2. *A dualmetric space is complete if and only if it is quasi-complete.*

Proposition 2. *A nuclear dualmetric space E which is quasi-complete is a Montel space.*

Proof. By Lemma 2, E is complete. Moreover E is barrelled because E is complete and quasi-barrelled (cf. [1]). Since E is a nuclear space, an arbitrary closed and bounded subset B is a closed and precompact subset. Hence B is compact because E is complete. Therefore E is a Montel space. Since a Montel space is reflexive, we have the following.

Corollary. *A nuclear dualmetric space E which is quasi-complete is reflexive.*

The following Lemma due to [7].

Lemma 3. *E is quasi-barrelled if and only if either of the following two equivalent conditions holds,*

(a) *The identity map from E_B onto E is almost open.*

(b) *E' is almost closed*) in E'_B and $E=E_\tau$, where E_B denote the associated bornological space (cf. [1, Ch. 3, § 2, Example 13]).*

By using the above result, we have the following theorem. The idea of its proof is essentially due to [7].

Theorem. *If E is nuclear dualmetric space which is quasi-complete, then E is the strong dual of Fréchet-Montel space.*

Proof. By Lemma 2, E is complete dualmetric space. Without loss of generality we can take a countable fundamental system of bounded set which is formed by closed bounded set because every bounded set is precompact in a nuclear space by Proposition 2 of S. Funakosi [4]. Therefore $E'^\beta = E'^k$, where E'^β (resp. E'^k) denotes the set E' with the topology of uniform convergence over the bounded (resp. compact) sets of E. First of all it will be shown that E'^β $(=E'^k)$

*) We say that F' is an almost closed subspace of E' if $U^\circ \cap F'$ is closed in E' for every neighborhood U of zero in E.

is a Fréchet space. Clearly E'^{β} is a metrizable space because E is a dualmetric space. It is sufficient therefore to deal with Cauchy sequence on E'^{β}. Let $\{x_n\}$ be a such a Cauchy sequence, and x be its limit in $E^{*\beta}$ $(=E^{*k})$, where E^* denotes the algebraic dual of E. It is easy to show that the restriction of the functional x onto a closed bounded subset of E is continuous because a closed bounded subset is a compact subset; in particular, x takes any convergent sequence in E into a convergent sequence of scalars. Therefore, x takes bounded sets into bounded sets of scalars. Thus $x \in E'_B$ (cf. [1. Ch. 3, § 2, Example 13]), and hence $\tilde{E}'^{\beta} \subset \tilde{E}'_B$, where $\tilde{E}'$ denotes the completion of the space E'. Since the set $\{x_n\} \cup \{x\}$ is compact in $\tilde{E}'^{\beta}$, its closed convex hull H will also be compact in $\tilde{E}'^{\beta}$ $(=\tilde{E}'^{k})$ and will therefore be compact as a subset of E'^{σ}_B, where E'^{σ} denotes the weak dual of E. Since nuclear dualmetric space is a quasi-barrelled, $H \cap E'$ is closed in E'^{σ}_B by Lemma 3. Hence this implies that x must be in $H \cap E' \subset E'$ along with $\{x_n\}$. Sequential completeness and therefore completeness of E'^{β} now follows. Moreover, since E is a Montel space E'^{β} is a Montel space (cf. [5. p. 32, Propopsition 17]). By Corollary of Proposition 1, topologies in $E=E'^{\beta'}$ $(=E'^{k'})$ are identical with the Mackey topology $\tau(E, E')$. Next, we establish that $E'^{\beta'k}=E'^{\beta'\tau}$ $(=E)$. In fact, the completeness of E'^{β} $(=E'^{k})$ ensures that $E'^{\beta'k}=E'^{\beta'c}$ where E'^{c} denotes the set E' with the topology of uniform convergence over the compact convex sets of E, and since a compact convex set of E'^{β} $(=E'^{k})$ is compact in the coarser topology $\sigma(E'^{\beta}, E)$, it follows that $E'^{\beta'k} \prec E'^{k'\tau}$. On the other hand, if K is a compact convex set in the weak topology $\sigma(E', E)$ and is therefore an equicontinuous subset of E', and as such, it is compact in E'^{β} $(=E'^{k})$ (cf. [1. Ch. 3, § 3, proposition 5]). This establishes the inverse inequality $E'^{\beta'k} \succ E'^{\beta'\tau}$, so that in fact $E'^{\beta'k}=E'^{\beta'\tau}$ $(=E)$. Finally E is the strong dual of the Fréchet-Montel space E'^{β}. The proof is as follows. The space E is a barrelled space because E is a complete dualmetric space. Since, however $E=E'^{\beta'k}=E'^{\beta'\tau}$ is the Mackey dual of E'^{β}, bounded subset of E'^{β} are relatively compact in the topology $\sigma(E', E)$, and as demonstrated in the proceding paragraph, such sets are relatively compact in E'^{β}. It follows therefore that $E=E'^{\beta'\tau}=E'^{\beta'\beta}$. The proof is complete.

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