180. *Note on Semigroups, which are Semilattices of Groups*

By S. LAJOS
K. Marx University, Budapest, Hungary

(Comm. by Kinjirô KUNUGI, M. J. A., Oct. 12, 1968)

Let S be a semigroup. Following the terminology of A. H. Clifford [1], [2] we say that S is a semilattice of groups if S is a set-theoretical union of a set $\{G_\alpha, \alpha \in I\}$ of mutually disjoint subgroups G_α such that, for every α, β in I, the products $G_\alpha G_\beta$ and $G_\beta G_\alpha$ are both contained in the same G_γ ($\gamma \in I$).

Recently the author proved the following characterization of semigroups, which are semilattices of groups (see [3]).

Theorem 1. *A semigroup S is a semilattice of groups if and only if*

(1) $$L_1 \cap L_2 = L_1 L_2$$

and

(2) $$R_1 \cap R_2 = R_1 R_2$$

for any two left ideals L_1, L_2 of S and right ideals R_1, R_2 of S, respectively.

In this note we give another characterization of semigroups, which are semilattices of groups.

Theorem 2. *A semigroup S is a semilattice of groups if and only if*

(3) $$L \cap A = LA$$

and

(4) $$R \cap A = AR$$

for any left ideal L, right ideal R, and two-sided ideal A of S.

Proof. *Necessity.* Let S be a semigroup which is a semilattice of groups. Then it is an inverse semigroup every one-sided ideal in which is a two-sided ideal (see [2]). This implies that

(5) $$A \cap B = AB$$

for any two ideals A, B of S. Therefore the relations (3), (4) are satisfied.

Sufficiency. Let S be a semigroup having the properties (3) and (4) for any left ideal L, right ideal R, and two-sided ideal A of S. In case of $A = S$ the equality (3) implies

(6) $$L \cap S = LS.$$

This means that any left ideal L is also a right ideal of S, whence L

is a two-sided ideal of S. Consequently the relation (1) is satisfied for any two left ideals L_1, L_2 of S.

Analogously, in case of $A=S$ the equality (4) implies that

$$R \cap S = SR, \tag{7}$$

that is, any right ideal R is a two-sided ideal of S. Therefore the relation (2) follows from (4). Finally Theorem 1 implies that S is a semilattice of groups.

Theorem 1 and Theorem 2 imply the result as follows.

Theorem 3. *For a semigroup S the following conditions are equivalent:*

(A) *S is a semilattice of groups.*

(B) *$L_1 \cap L_2 = L_1 L_2$ for any two left ideals L_1, L_2, and $R_1 \cap R_2 = R_1 R_2$ for any two right ideals R_1, R_2 of S.*

(C) *$L \cap A = LA$ and $R \cap A = AR$ for any left ideal L, right ideal R, and two-sided ideal A of S.*

References

[1] A. H. Clifford: Bands of semigroups. Proc. Amer. Math. Soc., **5**, 499–504 (1954).

[2] A. H. Clifford and G. B. Preston: The algebraic theory of semigroups. I, II. Amer. Math. Soc., Providence, R. I. (1961, 1967).

[3] S. Lajos: Characterizations of semigroups, which are semilattices of groups (in Hungarian). Magyar Tud. Akad. Mat. Fiz. Tud. Oszt. Közl. (to appear).