176. On the Sets of Points in the Ranked Space. III

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In this paper, for a subset A of a ranked space R [1], we shall define two subsets of the ranked space, $\bar{A}$ and $\tilde{A}$. Both of them have some properties which are analogous to the closure in the usual topological space. We shall introduce several propositions with respect to $\bar{A}$ and $\tilde{A}$. We have used the same terminology as that introduced in the paper "On the sets of points in the ranked space II." [6].

Definition. Let A be a subset of a ranked space R. Then $\bar{A}$ and $\tilde{A}$ are defined as follows.

$$\bar{A}=\{x\,;\,\exists\{V_\alpha(x)\},\ V_\alpha(x)\cap A\neq\phi \text{ for all } \alpha\},$$
$$\tilde{A}=\{x\,;\,\forall\{V_\alpha(x)\},\ V_\alpha(x)\cap A\neq\phi \text{ for all } \alpha\},$$

where $\{V_\alpha(x)\}$ is a fundamental sequence of neighborhoods with respect to a point x of R [2] and α is a natural number. We say that $\bar{A}$ is an r-closure of A and that $\tilde{A}$ is a quasi r-closure of A.

Proposition 1. *If A is a subset of a ranked space R, then*

(1) $\tilde{A}\subseteq\bar{A}$,

(2) *if R satisfies Condition* (M) [3] *then* $\bar{A}=\tilde{A}$.

Proof. It is easy to prove (1).

If $p\in\bar{A}$, then by the definition there exists a fundamental sequence of neighborhoods of p, $\{V_\alpha(p)\}$, such that $V_\alpha(p)\cap A\neq\phi$ for all α.

Let $\{U_\beta(p)\}$ be an arbitrary fundamental sequence of neighborhoods of p, and $V_\alpha(p)\in\mathfrak{U}_{\gamma_\alpha}$ and $U_\beta(p)\in\mathfrak{U}_{\delta_\beta}$. Then for each β, there exists γ_α such that $\delta_\beta\leq\gamma_\alpha$. By Condition (M), $U_\beta(p)\supseteq V_\alpha(p)$, consequently $U_\beta(p)\cap A\neq\phi$. Therefore $p\in\tilde{A}$, that implies $\bar{A}\subseteq\tilde{A}$. Then, $\bar{A}=\tilde{A}$ because by (1) $\bar{A}\supseteq\tilde{A}$.

Remark 1. In general $\bar{A}\neq\tilde{A}$. For example, if $A=\{z_n\}$, where $\{z_n\}$ is a sequence of points in Example 1 [3], then $\bar{A}\neq\tilde{A}$.

Proposition 2. *If A and B are subsets of a ranked space, then*

(1) *if $A\subseteq B$, then $\bar{A}\subseteq\bar{B}$ and $\tilde{A}\subseteq\tilde{B}$,*

(2) *$A\subseteq\bar{A}$ and $A\subseteq\tilde{A}$,*

(3) *$\overline{A\cup B}=\bar{A}\cup\bar{B}$ and $\widetilde{A\cup B}=\tilde{A}\cup\tilde{B}$,*

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(4) $\bar{\phi}=\phi$ *and* $\tilde{\phi}=\phi$,

(5) $\bar{R}=\bar{R}$ *and* $\tilde{R}=R$.

Proof. It is easy to prove this proposition.

Remark 2. $\bar{A}=\bar{\bar{A}}$ and $\tilde{A}=\tilde{\tilde{A}}$ are not always true. For example, let A be a point p in the example of K. Kunugi [2]. Then, it is shown that $\bar{A}$ is a proper subset of $\bar{\bar{A}}$ and that $\tilde{A}$ is a proper subset of $\tilde{\tilde{A}}$.

Proposition 3. *If A is a subset of a ranked space R, then the following conditions are equivalent.*

(a) *A is an r-closed subset of R.*

(b) $\bar{A}=A$.

Proof. First we will prove that (a) implies (b).

Suppose that $A\neq\bar{A}$, that is, $A\not\supseteq\bar{A}$. Then there exists a point p such that $p\in\bar{A}$ and $p\notin A$. Consequently, $p\in R-A$ and there exists a fundamental sequence of neighborhoods of p, $\{V_\alpha(p)\}$, such that $V_\alpha(p)\cap A\neq\phi$ for all α. Hence $R-A$ is not an r-open subset of R. Therefore A is not an r-closed subset of R.

Next we will prove that (b) implies (a).

If A is not an r-closed subset, $R-A$ is not an r-open subset of R. Therefore, there exist a point p of $R-A$ and a fundamental sequence of neighborhoods of p, $\{V_\alpha(p)\}$, such that $V_\alpha(p)\cap A\neq\phi$ for all α. Hence $p\in\bar{A}$. Consequently $A\neq\bar{A}$ because $p\notin A$.

Proposition 4. *If A is a subset of a ranked space R, then the conditions below are related as follows. For all spaces the condition* (a) *implies the condition* (b), *but the converse is not always true.*

(a) *A is an r-closed subset of R.*

(b) $\tilde{A}=A$.

Proof. If A is an r-closed subset of R, then $A=\bar{A}$ by Proposition 3. Since $A\subseteq\tilde{A}\subseteq\bar{A}$, we have $A=\tilde{A}$.

The example of Remark 1 shows that the converse is not always true, because $A=\tilde{A}$ and $A\neq\bar{A}$.

Proposition 5. *Let $\{p_\alpha\}$ be an arbitrary sequence of a ranked space R and $A_\beta=\{p_\beta, p_{\beta+1}, \cdots\}$, $(\beta=1, 2, \cdots)$, then the following conditions are equivalent.*

(a) *When p is a point of R, $p\in\bar{A}_\beta$ for all β.*

(b) *A point p is an r-cluster point of $\{p_\alpha\}$.*

Proof. First we will prove that (a) implies (b).

If a point p is not an r-cluster point of $\{p_\alpha\}$, then for each fundamental sequence of neighborhoods of p, $\{V_\alpha(p)\}$, and for each natural number γ such that $\beta\leq\gamma$, there exists a natural number β and $V_{\alpha_0}(p)$ such that $p_\gamma\notin V_{\alpha_0}(p)$. Hence $V_{\alpha_0}(p)\cap A_\beta=\phi$. By the condition (a), there exists a fundamental sequence of neighborhoods of p, $\{U_\alpha(p)\}$, such that $U_\alpha(p) \cap A_\beta \neq \phi$ for all α. This is a contradiction.

Next we will prove that (b) implies (a).

Since p is an r-cluster point of $\{p_\alpha\}$, there exists a fundamental sequence of neighborhoods of p, $\{V_\alpha(p)\}$, such that $\{P_\alpha\}$ is frequently in each $V_\alpha(p)$. Consequently, for each $V_\alpha(p)$ and an arbitrary natural number β, there exists $\delta(\alpha)$ such that $\beta < \delta(\alpha)$ and $p_{\delta(\alpha)} \in V_\alpha(p)$. Since $p_{\delta(\alpha)} \in A_\beta$, we have $V_\alpha(p) \cap A_\beta \neq \phi$ for all α. Hence $p \in \bar{A}_\beta$ for all β.

Proposition 6. *Let $\{p_\alpha\}$ be an arbitrary sequence of a ranked space R and $A_\beta = \{P_\beta, p_{\beta+1}, \cdots\}$, $(\beta = 1, 2, \cdots)$, then the conditions below are related as follows. For all spaces the condition* (a) *implies the condition* (b), *but the converse is not always true.*

(a) *When p is a point of R, $p \in \tilde{A}_\beta$ for all β.*

(b) *A point p is an r-cluster point of $\{p_\alpha\}$.*

Proof. Let $\{V_\alpha(p)\}$ be an arbitrary fundamental sequence of neighborhoods of p. Since $p \in \tilde{A}_\beta$, we have $V_\alpha(p) \cap A_\beta \neq \phi$ for all β. Therefore, $\{p_\alpha\}$ is frequently in each neighborhood $V_\alpha(p)$. Hence p is an r-cluster point of $\{p_\alpha\}$.

The example of Remark 1 shows that the converse is not always true. For example, let p_α be z_α in the example of Remark 1, then the r-cluster point p of the sequence $\{p_\alpha\}$ does not belong to $\tilde{A}_\beta$ for all β.

Proposition 7. *If R is a ranked space, then the conditions below are related as follows. For all spaces the condition* (a) *implies the condition* (b), *but the converse is not always true.*

(a) *If quasi r-closures $\tilde{B}_\alpha$ of subsets B_α $(\alpha = 1, 2, \cdots)$ in R are non-empty subsets of R and $\tilde{B}_1 \supseteq \tilde{B}_2 \supseteq \cdots \supset \tilde{B}_\alpha \supseteq \cdots$, then $\bigcap_\alpha \tilde{B}_\alpha \neq \phi$.*

(b) *R is a sequentially compact set.*

Proof. Let $\{p_\alpha\}$ be an arbitrary sequence of R and $B_\alpha = \{p_\alpha, p_{\alpha+1}, \cdots\}$, $(\alpha = 1, 2, \cdots)$. We have $\tilde{B}_1 \supseteq \tilde{B}_2 \supseteq \cdots \supseteq \tilde{B}_\alpha \supseteq \cdots$ and $\tilde{B}_\alpha \neq \phi$. Consequently, by the hypothesis there exists a point p such that $p \in \tilde{B}_\alpha$ for all α. By Proposition 6, p is an r-cluster point of $\{p_\alpha\}$. Hence R is a sequentially compact set.

The following example shows that the converse is not always true.

Let us consider the ranked space E of Example 2 [2]. Let $R = \{z_n\} \cup \{p\}$, and $U = R \cap V$, where V is a neighborhood of a point in E. If the rank of U is defined to be that of V, R becomes a ranked space. Then R is a sequentially compact set. However, if we suppose that $B_\alpha = \{Z_\alpha, z_{\alpha+1}, \cdots\}$, $(\alpha = 1, 2, \cdots)$, the condition (a) is not satisfied.

Proposition 8. *If R is a ranked space, then the following conditions are equivalent.*

(a) *If B_α are non-empty r-closed subsets of R, and* $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_\alpha \supseteq \cdots$, *then* $\bigcap_\alpha B_\alpha \neq \phi$.

(b) *R is a sequentially compact set.*

Proof. First we will prove that (a) implies (b).

Since B_α is an r-closed subset, $B_\alpha = \tilde{B}_\alpha$. Therefore, by Proposition 7, R is a sequentially compact set.

Next we will prove that (b) implies (a).

Since $B_\alpha \neq \phi$ there exists a sequence $\{p_\alpha\}$ such that $p_\alpha \in B_\alpha$ for all α. Suppose that $C_\alpha = \{p_\alpha, p_{\alpha+1}, \cdots\}$ $(\alpha = 1, 2, \cdots)$. Since R is a sequentially compact set, $\{p_\alpha\}$ has an r-cluster point p. By Proposition 5, $p \in \bar{C}_\alpha$ $(\alpha = 1, 2, \cdots)$. Noting that B_α is an r-closed subset of R, $\bar{B}_\alpha = B_\alpha$. Hence $p \in B_\alpha$ for all α. Consequently, $\bigcap_\alpha B_\alpha \neq \phi$.

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