173. *An Axiomatic Characterization of the Large Inductive Dimension for Metric Spaces*

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An axiomatic characterization of dimension was given by Menger for subsets of Euclidean plane in 1929 [1]. Recently, Nishiura generalized Menger's result in the following form [4]:

Suppose that f is an extended real-valued function on the collection of separable metrizable spaces. Then f is the dimension function if and only if f satisfies the following seven conditions:

(a) *f is topological*; that is, X homeomorphic to Y implies $f(X)=f(Y)$.

(b) *f is monotone*; that is, $X\subset Y$ implies $f(X)\leqq f(Y)$.

(c) *f is F_σ-constant*; that is, $X=\bigcup_{i=1}^{\infty}X_i$, where X_i is a closed subset of X, $i=1, 2, \cdots$, implies $f(X)\leqq \sup f(X_i)$.

(d) *f is inductively subadditive*; that is, $X=A\cup B$ implies $f(X)\leqq f(A)+f(B)+1$.

(e) *f is compactifiable*; that is, each space X is homeomorphic to a subspace of a compact space Y for which $f(X)=f(Y)$.

(f) *f is pseudo-inductive*; that is, for each space X and $x\in X$ there are arbitrarily small neighbourhoods U of x such that $f(\bar{U}-U)\leqq f(X)-1$. (We agree that $\infty-1=\infty$.)

(g) $f(\{\phi\})=0$.

Furthermore, the seven conditions are independent.

The purpose of the present paper is to give a generalization of Nishiura's result for metrizable spaces without separability. For any metric space X we have $\operatorname{Ind} X=\dim X$ (cf. [2] or [3]), but there exists a complete metric space X with $\operatorname{ind} X=0$ and $\operatorname{Ind} X=1$ [5]. In view of the fact that the large inductive dimension has many remarkable properties besides the property mentioned above, this paper will be devoted to the study of large inductive dimensions.

Before stating the theorem, we fix some terminology. For the definitions of ind (the small inductive dimension), Ind (the large inductive dimension) and dim (the covering dimension), see [3]. Spaces $X, Y, \cdots$ will be assumed to be metrizable spaces, and dim will be used both for the large inductive dimension and for the covering dimension. Let Ω be a non-empty set. Generalized Baire space $N(\Omega)$

is made up by all sequences of elements from Ω, and for any two sequences $a=(a_1, a_2, \cdots)$, $b=(b_1, b_2, \cdots)$ of $N(\Omega)$ we define $\rho(a, b)$ as follows:

$$\rho(a, b)=1/k \text{ if } a_i=b_i \text{ for } i<k \text{ and } a_k\neq b_k,$$
$$\rho(a, b)=0 \quad \text{ if } a_i=b_i \text{ for } i=1, 2, \cdots.$$

As was observed in [2], $N(\Omega)$ is a complete metric space and $\dim N(\Omega)=0$. For an ordinal number α we denote $N(W(\alpha))$ by $N(\alpha)$ simply, where $W(\alpha)=\{\gamma \mid \gamma<\alpha, \gamma \text{ is ordinal}\}$.

Now, employing the terminology used by Nishiura, our characterization is stated as follows:

Theorem. *Let f be an extended real-valued function on the family of all metrizable spaces. In order that f be the large inductive dimension it is necessary and sufficient that f satisfies the following five conditions:*

(1) *f is topological.*

(2) *f is monotone.*

(3) *f is inductively subadditive.*

(4) *f is pseudo-inductive*[1]; that is, for each space X, a closed subset F of X and an open subset G of X with $F\subset G$, there exists an open subset U of X such that $F\subset U\subset G$ and $f(\bar{U}-U)\leqq f(X)-1$.

(5) *f is normed*; that is, for each non-zero ordinal number α, $f(N(\alpha))=0$.

Furthermore, these five conditions are independent.

Proof. Clearly, dim satisfies the conditions (1)-(5). We prove the converse in five parts. Suppose that f satisfies the five conditions of the theorem.

(I) $f(\{\phi\})=0$.

Proof. If α in the condition (5) is 1, then $\{\phi\}$ is homeomorphic to $N(\alpha)$ and by the conditions (1) and (5) we have $f(\{\phi\})=0$.

Thus, by the same method as in [4], the following assertions (II) and (III) are proved. To make our proof self-contained, we give their proofs.

(II) $f(X)=-1$ *if and only if* $X=\phi$.

Proof. (I) and the condition (4) imply $f(\phi)\leqq -1$. By (I) and the condition (3) we have

$$0=f(\{\phi\})=f(\{\phi\}\cup\phi)\leqq f(\{\phi\})+f(\phi)+1=f(\phi)+1.$$

Hence $f(\phi)=-1$. If $X\neq\phi$, we have $f(X)\geqq f(\{\phi\})=0>-1$ by the conditions (1), (2), and the assertion (I). Therefore, the assertion (II) is proved.

(III) *For each extended integer n ($n\geqq -1$),*

$$f(X)\leqq n \quad \textit{implies} \quad \dim X\leqq n.$$

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1) This condition is an extension of Nishiura's (f).

Proof. We carry out the proof by induction. The proposition is true for $n=-1$ by (II) and the condition (2). Suppose that the proposition is true for n ($n<\infty$) and that $f(X)\leqq n+1$. By the condition (4), for each closed set F and open set G in X with $F\subset G$ there exists an open set U in X such that $F\subset U\subset G$ and $f(\bar{U}-U)\leqq n$. By the induction hypothesis, $\dim(\bar{U}-U)\leqq n$ and hence $\dim X\leqq n+1$. Thus the induction is completed.

(IV) *For each extended integer n ($n\geqq -1$),*

$$\dim X\leqq n \quad \textit{implies} \quad f(X)\leqq n.$$

Proof. The proposition is true for $n=-1$ by (II). Suppose $\dim X=0$. It was shown by Morita that X is homeomorphic to a subspace of $N(\Omega)$, where Ω is a set whose cardinal number is not less than the cardinal number of a basis of open sets of X [2, Theorem 10.2]. Thus by the conditions (1), (2), and (5), we have $f(X)\leqq 0$. Now, let $\dim X\leqq n<\infty$. Then by the decomposition theorem of dim [2, Theorem 5.3], there are subspaces $X_1, X_2, \cdots, X_{n+1}$ of X with $X=\bigcup_{i=1}^{n+1} X_i$ and $\dim X_i\leqq 0$, $i=1, 2, \cdots, n+1$. Hence, by the fact proved above we have $f(X_i)\leqq 0$, $i=1, 2, \cdots, n+1$. Thus by the condition (3), we have $f(X)\leqq n$, and (IV) is proved.

(V) $f(X)=\dim X$ *for all X.*

Proof. This follows from the assertions (III) and (IV).

Thus the first part of our theorem is proved. We prove the second part by giving examples. Example i does satisfy the five conditions except the condition (i), $i=1, 2, \cdots, 5$. Except Example 2, the verifications are straightforward and are omitted.

Example 1. Let f be defined as follows: $f(X)=\dim X$ if X is a subspace of $N(\alpha)$ for some ordinal number α; $f(X)=\dim X+1$ if X is not a subspace of $N(\alpha)$ for any ordinal number α.

Example 2. Let f be defined as follows: $f(X)=\dim X$ if X is a complete n-dimensional space; $f(X)=\infty$ if X is not complete.

Proof. Let $h: X\rightarrow Y$ be a homeomorphism and let d be a complete metric on X. If we put

$$\rho(y_1, y_2)=d(h^{-1}(y_1), h^{-1}(y_2)) \text{ for } y_1, y_2\in Y,$$

then ρ is a complete metric on Y compatible with the topology of Y. Since $\dim X=\dim Y$, f satisfies the condition (1).

Let $X=A\cup B$, where A and B are complete subspaces of X. Suppose that X is embedded in a complete space Y. Then, A and B are G_δ-sets in Y, that is, $A=\bigcap_{i=1}^{\infty} A_i$, $B=\bigcap_{i=1}^{\infty} B_i$, where A_i and B_i are open subsets of Y, $i=1, 2, \cdots$. Then $X=\bigcap_{i=1}^{\infty}(A_i\cup B_i)$ and $A_i\cup B_i$ is open in Y, $i=1, 2, \cdots$. Therefore, X is a G_δ-set in Y and X is complete. Since $\dim X \leqq \dim A + \dim B + 1$, f satisfies the condition (3).

Proofs of the other parts are easy and are omitted.

Example 3.[2] Let f be defined as follows: $f(X) = \dim X$ if $\dim X \leqq 0$; $f(X) = \dim X + 1$ if $\dim X > 0$.

Example 4. Let f be defined as follows: $f(\phi) = -1$; $f(X) = \dim X / (\dim X + 1)$ if $-1 < \dim X < \infty$; $f(X) = 1$ if $\dim X = \infty$.

Example 5. Let f be defined as follows: $f(X) = \dim X + 1$ for all X.

Thus the independence of the conditions (1)-(5) in the theorem is established, and the theorem is proved.

References

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2) Examples 3, 4, and 5 are due to Nishiura [4].