172. Semigroups Satisfying $xy^m = yx^m = (xy^m)^n$

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(Comm. by Kenjiro SHODA, M. J. A., Oct. 12, 1968)

Recently E. J. Tully [5] determined the semigroups satisfying an identity of the form $xy = y^m x^n$; Tamura [4], one of the authors, studied the semigroups satisfying an identity $xy = y^{m_1}x^{n_1}\cdots y^{m_k}x^{n_k}$; and Mead [2], the other author, found a necessary and sufficient condition in order that an implication, $x^n y^m = y^k x^l \rightarrow x^n y^m = y^n x^m$, hold in all semigroups. Related to these works the purpose of this paper is to find the structure of semigroups satisfying an identity of the form

$$(*) \qquad xy^m = yx^m = (xy^m)^n, \qquad n > 1.$$

Let L be a semilattice and $\{S_\alpha : \alpha \in L\}$ be a family of disjoint semigroups. If a semigroup S is a union of disjoint subsemigroups S'_α, $\alpha \in L$, and if S'_α is isomorphic with S_α for all α and $S'_\alpha S'_\beta \subseteq S'_{\alpha\beta}$ for all $\alpha, \beta \in L$, then S is called a semilattice-union of S_α, $\alpha \in L$, or a semilattice of S_α, $\alpha \in L$. A semigroup S is called a Clifford semigroup if S is a union of groups.

Lemma. *A Clifford semigroup S is commutative if and only if S is a semilattice-union of abelian groups.*

Proof. S is a semilattice-union of completely simple semigroups S_α by Theorem 4.6 [1]. Since S is commutative, each S_α is an abelian group. The converse is obtained from Theorem 4.11 [1].

Let I be an ideal of a semigroup S and $S/I \cong Z$. Then S is called an ideal extension of I by Z.

Theorem. *The following three statements are equivalent.*

(1) *A semigroup S satisfies the identity $(*)$.*

(2) *A semigroup S contains a commutative Clifford subsemigroup M and satisfies*

(2.1) $x^{k+1} = x$ *for all $x \in M$, where k is the greatest common divisor of $m-1$ and $n-1$.*

(2.2) $xy^m \in M$ *for all $x, y \in S$.*

(3) *A semigroup S is a semilattice-union of semigroups S_α, $\alpha \in L$, such that each S_α is an ideal extension of a group G_α by Z_α and the following conditions are satisfied:*

(3.1) *Each G_α is abelian and satisfies $x^k = e$ for all $x \in G_\alpha$, where e is the identity element of G_α, k being defined in* (2.1).

(3.2) *Z_α satisfies $xy^m=0$ for all $x, y \in Z_\alpha$.*

(3.3) *If $x_\alpha \in S_\alpha$ and $y_\beta \in S_\beta$, $\alpha \neq \beta$, then $x_\alpha y_\beta^m \in G_{\alpha\beta}$.*

Proof. (1)→(2). Suppose that a semigroup S satisfies the identity $(*)$. Let $M=\{xy^m : x, y \in S\}$. M is a subsemigroup of S and $z=z^n$, $n>1$, for all $z \in M$, by $(*)$. Since every element of M is of index 1, M is a union of groups (Ex. 1.7, 6(a), p. 23 [1]), hence M is regular (p. 26 [1]). Also by $(*)$ any two idempotents of M commute. Therefore S is an inverse semigroup by Theorem 1.17, [1]. According to Ex. 4.2, 2, p. 129 [1], M is a semilattice-union of groups, say

$$M=\bigcup_{\alpha\in L} G_\alpha.$$

The identity $(*)$ in the groups G_α turns out to be

$$x=x^m=x^n \text{ and } xy=yx \text{ for all } x, y \in G_\alpha,$$

that is, G_α is abelian and satisfies (2.1). By the Lemma the Clifford semigroup M is commutative. (2.2) is clear by the definition of M.

(2)→(3). Assume (2), $M=\bigcup_{\alpha\in L} G_\alpha$, G_α abelian groups. By (2.1) and (2.2) there are positive integers l such that x^l are idempotent for all $x \in S$. For example $l=(m+1)(n-1)$. First we notice that

(4) if e is any idempotent, $eze=ze$ for all $z \in S$

since e, $ze \in M$ by (2.2) and M is commutative. Let $x^l=e$, $y^l=f$, and $(xy)^l=h$ where e, f, h are idempotents. To prove $h=ef$,

$$\begin{aligned} h=(xy)^l=(xy)^l h &= \{(xh)(yh)\}^l && \text{by (4)} \\ &=(xh)^l(yh)^l && \text{by commutativity of } M \\ &=(x^l h)(y^l h) && \text{by (4)} \\ &=(eh)(fh) && \\ &=efh && \text{by (4)} \end{aligned}$$

and $ef=x^l y^l=x^l y^l f=x^l f y^l f=(xy)^l f=hf$ by the same reason. Hence $h=efh$ and $ef=hf$. Since the idempotents from a semilattice $h=efh=hfh=hf=ef$. Consequently we have

(5) $$(xy)^l=x^l y^l$$

that is, the mapping $x\to x^l$ is a homomorphism of S onto the semilattice L_1 of all idempotents of S. Clearly $L_1 \subseteq M$ and L_1 is the set of identity elements of G_α, $\alpha \in L$; hence $L_1 \cong L$, so we identify L_1 with L. Let e_α be the identity element of G_α. We define S_α by

$$S_\alpha=\{x \in S : x^l=e_\alpha\}.$$

Then $G_\alpha \subseteq S_\alpha$ and

(6) $$S=\bigcup_{\alpha\in L} S_\alpha.$$

Each S_α is unipotent, i.e., has a unique idempotent e_α, and S_α is inversible in the sense of [3], and it is easily seen that

$$S_\alpha e_\alpha=G_\alpha$$

hence G_α is an ideal of S_α (see [3]). The condition (3.1) is obvious by the assumption; (3.2) and (3.3) are obtained by (2.2).

(3)→(1). Assume (3) and let $S=\bigcup_{\alpha\in L} S_\alpha$ and $M=\bigcup_{\alpha\in L} G_\alpha$. Since G_α is abelian, M is commutative by the Lemma. By (3.2) and (3.3) $x_\alpha y_\beta^m \in G_{\alpha\beta}$ for all $x_\alpha \in S_\alpha$, $y_\beta \in S_\beta$. It follows from (3.1) that $x_\alpha y_\beta^m = (x_\alpha y_\beta^m)^n$. We need to prove $x_\alpha y_\beta^m = y_\beta x_\alpha^m$. Both $x_\alpha y_\beta^m$ and $y_\beta x_\alpha^m$ are in $G_{\alpha\beta}$. Since M is commutative and e, $ze \in M$, we can apply (4) to the present case again. Using (4) and (3.1)

$$\begin{aligned} x_\alpha y_\beta^m &= x_\alpha y_\beta^m e_{\alpha\beta} = x_\alpha e_{\alpha\beta}(y_\beta e_{\alpha\beta})^m = (x_\alpha e_{\alpha\beta})(y_\beta e_{\alpha\beta}) \\ &= (y_\beta e_{\alpha\beta})(x_\alpha e_{\alpha\beta}) = (y_\beta e_{\alpha\beta})(x_\alpha e_{\alpha\beta})^m = y_\beta x_\alpha^m. \end{aligned}$$

This completes the proof of the theorem.

Remark 1. We can prove directly (1)→(3) by means of (5), and the minimum l_0 of l's which act in the proof of (2)→(3) is determined as follows:

l_0 is the minimum of the positive integers greater than or equal to $m+1$ and divisible by k.

Remark 2. M is a left ideal of S but need not be an ideal.

Example. The semigroup S defined by the Cayley table:

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	b	c	d

S satisfies the identity $xy^2=yx^2=(xy^2)^2$ and $M=\{a, d\}$ is a left ideal but not a right ideal.

References

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