171. Notes on Medial Archimedean Semigroups without Idempotent

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1. Introduction. A semigroup S is called medial if S satisfies the identity $xyzu=xzyu$. According to Chrislock [1], [2], a medial semigroup S is $\mathcal{S}$-indecomposable (or $\mathfrak{p}$-simple), that is, having no semilattice-homomorphic image except a trivial one, if and only if S satisfies: for every $a, b \in S$ there are $x, y, z, u \in S$ and positive integers m and n such that

$$a^m=xby \quad \text{and} \quad b^n=zau.$$

This property is called archimedeaness which coincides with "archimedeaness" [3] in commutative semigroups.

The author proved the following theorem (cf. [4]):

Theorem 1. *If S is a commutative archimedean semigroup without idempotent, the closet*[1] *is empty for all elements, that is,*

$$(1) \qquad \bigcap_{n=1}^{\infty} a^nS=\phi \quad \text{for all } a \in S.$$

In this note we will extend this theorem to medial semigroups and will state its various applications.

Theorem 2. *If S is a medial archimedean semigroup without idempotent, then*

$$(2) \qquad \bigcap_{n=1}^{\infty} Sa^nS=\phi \quad \text{for all } a \in S.$$

Proof. Let $D=\bigcap_{n=1}^{\infty} Sa^nS$ and suppose that $D\neq\phi$. Then $aDa\neq\phi$ for all $a \in S$. By mediality

$$(3) \qquad aDa\subseteq \bigcap_{n=1}^{\infty} aSa^nSa=\bigcap_{n=1}^{\infty} a^naS^2a\subseteq \bigcap_{n=1}^{\infty} a^naSa=\bigcap_{n=1}^{\infty} a^{3m}aSa.$$

On the other hand, aSa is obviously a subsemigroup and it is commutative since

$$(axa)(aya)=(aya)(axa) \quad \text{for all } x, y \in S.$$

We will prove that aSa is archimedean. Since S is medial archimedean, for axa and aya, there are $u, v \in S$, and a positive integer k such that

$$(axa)^k=u(aya)v.$$

1) $\bigcap_{n=1}^{\infty} a^nS$ is called the closet of a. See [5].

Then

$$(axa)^{k+2}=(axa)u(aya)v(axa)=(axua)(aya)(avxa)$$

by mediality. This shows that aSa is archimedean in the medial sense, hence archimedean in the commutative sense. Since S has no idempotent, aSa has no idempotent. By Theorem 1,

$$\bigcap_{m=1}^{\infty}(a^3)^m aSa=\phi.$$

Therefore aDa has to be empty; this is a contradiction to $aDa\neq\phi$. Thus we have proved that $D=\phi$.

Remark. It is easy to see that (1) and (2) are equivalent if S is commutative. If S is commutative, $\bigcap_{n=1}^{\infty} a^nS=\bigcap_{n=1}^{\infty} Sa^nS$. Even if S is not commutative we define the closet $C(a)$ of a by $C(a)=\bigcap_{n=1}^{\infty} Sa^nS$.

2. Application. Corollary 3. *A medial simple semigroup contains at least one idempotent.*

Proof. A simple semigroup S is $\mathcal{S}$-indecomposable, hence archimedean. Suppose S has no idempotent. By Theorem 2, $\cap\, Sa^nS=\phi$ for all $a\in S$. On the other hand, simpleness implies $Sa^nS=S$ for all $a\in S$, hence $\cap\, Sa^nS=S$. This is a contradiction. Therefore S contains an idempotent.

Theorem 4. *A semigroup S is medial and simple if and only if S is isomorphic onto the direct product of an abelian group and a rectangular band. Accordingly S is completely simple.*

Proof. Chrislock proved in his thesis [1], [2] that if S is medial and simple and if S contains idempotents, the conclusion of Theorem 4 is true. Corollary 3 reduces Theorem 4 to his result.

3. General remark. We have proved Theorem 4 by using Theorem 2. On the other hand, assuming Theorem 4 we can easily prove Theorem 2. The equivalence of these can be stated in more general form.

A semigroup S is called concentric if the closet $C(a)=\bigcap_{n=1}^{\infty} Sa^nS$ is constant, i.e., independent of a. We notice that $C(a)$ could be empty.

Theorem 5. *Let $\mathfrak{P}$ be a class of concentric semigroups and suppose $\mathfrak{P}$ satisfies*

(4) *If $S\in\mathfrak{P}$, an ideal of S is in $\mathfrak{P}$.*

Under (4), *the following two conditions on $\mathfrak{P}$ are equivalent*:

(5) *If $S\in\mathfrak{P}$ and if S has no idempotent, then*

$$C(a)=\phi \quad \text{for all } a\in S.$$

(6) *If $S\in\mathfrak{P}$ and if S is simple, S contains at least one idempotent.*

Proof. (6)$\rightarrow$(5): Suppose S has no idempotent and $C(a)\neq\phi$. Since S is concentric, it is easily proved that $C(a)$ is the minimal ideal of S, hence $C(a)$ is simple. By (4), $C(a) \in \mathfrak{P}$. Therefore $C(a)$ has an idempotent by (6). This is a contradiction. Conversely (5)→(6): The same proof as given in Corollary 3.

An identity of the form $x_1 x_2 \cdots x_n = x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(n)}$ in which π is a permutation is called a permutation identity. A semigroup is called quasi-commutative if it satisfies a non-trivial permutation identity. This terminology is due to Miyuki Yamada. Peter Perkins proved in his unpublished paper

(7) If a semigroup S is quasi-commutative and if $S^2 = S$, then S is medial.

Immediately Theorem 4 can be extended to quasi-commutative semigroups. Let $\mathfrak{P}_1$ be the class of all quasi-commutative $\mathfrak{p}$-simple semigroups. Then $\mathfrak{P}_1$ satisfies (6) and (4). The following question, however, is still open:

Is a quasi-commutative $\mathfrak{p}$-simple semigroup concentric?

Addendum. We notice that Professor Miyuki Yamada recently proved Theorem 4 from the standpoint of inversive semigroups and Professor J. L. Chrislock also proved Theorem 4 independently of this paper. Professor Naoki Kimura proved (7) in a simple way according to his personal letter to the author.

References

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