167. *On the Absolute Convergence of Fourier Series*

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1. The following theorems are due to Izumi [2]:

Theorem A. *Let* $f(t)\sim\sum_{1}^{\infty}a_n\cos nt$. *If*

(i) $\int_0^\pi \log\frac{2\pi}{t}|df(t)|<\infty$ *and* (ii) $\{n^\delta\Delta(na_n)\}\in BV$

for some $\delta>0$, *then* $\sum|a_n|<\infty$.

Theorem B. *Let* $g(t)\sim\sum_{1}^{\infty}b_n\sin nt$. *If*

(i)* $\int_0^\pi \log\frac{2\pi}{t}|dg(t)|<\infty$ *and* (ii)* $\{n^\delta\Delta(nb_n)\}\in BV$

for some $\delta>0$, *then* $\sum|b_n|<\infty$.

Theorem C. *Let* $f(t)\sim\sum_{1}^{\infty}a_n\cos nt$. *If*

(i)′ $f(t)\in BV(0,\pi)$ *and* (ii)′ $\{n^\delta\Delta(na_n)\}\in BV$
for some $\delta>0$, *then* $\sum|a_n|/\log n<\infty$.

Theorem D. *Let* $f(t)\sim\sum_{1}^{\infty}a_n\cos nt$ *and let* $\alpha>\beta+2$ *and* $\beta>0$. *If*

(i)″ $\int_0^\pi t^{-1/\beta}|df(t)|$ *and* (ii)″ $\{(\log n)^\alpha\Delta(na_n)\}\in BV$,

then $\sum|a_n|<\infty$.

In this note the following theorems will be established which are generalizations of the results mentioned above:

Theorem 1. *Let* $f(t)\sim\sum_{1}^{\infty}a_n\cos nt$. *If*

$$\int_0^\pi \log\frac{k}{t}|df(t)|<\infty \tag{1.1}$$

and

$$\left\{\frac{1}{e^{n^\alpha}}\sum_{1}^{n}e^{v^\alpha}a_v\right\}\in BV,\qquad 0<\alpha<1, \tag{1.2}$$

then $\sum|a_n|<\infty$.

Theorem 2. *Let* $g(t)\sim\sum_{1}^{\infty}b_n\sin nt$ *with* $g(+0)=0$. *If*

$$\int_0^\pi \log\frac{k}{t}|dg(t)|<\infty \tag{1.3}$$

and (1.2) *holds, then* $\sum|b_n|<\infty$.

Theorem 3. *Let* $f(t)\sim\sum_1^\infty a_n\cos nt$. *If*

$$f(t)\in BV(0,\pi) \tag{1.4}$$

and

$$\left\{\frac{1}{e^{n^\alpha}}\sum_1^n\frac{a_v e^{v^\alpha}}{\log(v+1)}\right\}\in BV,\qquad 0<\alpha<1, \tag{1.5}$$

then

$$\sum_1^\infty\frac{|a_n|}{\log(n+1)}<\infty.$$

Theorem 4. *If* $f(t)\sim\sum_1^\infty a_n\cos nt$ *and*

$$\int_0^\pi t^{-\gamma}|df(t)|<\infty, \tag{1.6}$$

$$\left\{\frac{1}{e^{n(\log n)^{-\delta}}}\sum_2^n a_v e^{v(\log v)^{-\delta}}\right\}\in BV \tag{1.7}$$

where $\delta=1+1/\gamma$ *and* $\gamma>0$, *then* $\sum|a_n|<\infty$.

2. The following lemmas will be required for the proof of our theorems:

Lemma 1 [3]. *If* $\{c_m\}\in BV$, *then* $\left\{\frac{1}{\lambda_m}\sum_1^m\mu_n c_n\right\}\in BV$, *where* $\lambda_m=\sum_1^m\mu_n$, $\mu_n>0$.

Lemma 2. *If the sequence* $\{n^\delta\Delta(na_n)\}\in BV$ *for some* $\delta>0$, *then* $\left\{\frac{1}{e^{n^\alpha}}\sum_1^n a_m e^{m^\alpha}\right\}\in BV$, *where* $0<\alpha<1$.

Proof. There is no loss of generality in assuming that $0<\delta<1$ and $\alpha=1-\delta$. Now

$$\begin{aligned}\frac{1}{e^{n^\alpha}}\sum_1^n a_r e^{r^\alpha}&=\frac{1}{e^{n^\alpha}}\sum_1^n ra_r\frac{e^{r^\alpha}}{r}\\&=\frac{1}{e^{n^\alpha}}\sum_1^{n-1}\Delta(ra_r)\sum_1^r\frac{e^{k^\alpha}}{k}+\frac{na_n}{e^{n^\alpha}}\sum_1^n\frac{e^{r^\alpha}}{r}\\&=L_1+L_2,\ \text{say.}\end{aligned}$$

Since $\left\{\frac{1}{e^{n^\alpha}}\sum_1^{n-1}\frac{e^{r^\alpha}}{r^\delta}\right\}\in BV$ it follows by virtue of Lemma 1 that

$$\left\{\frac{1}{e^{n^\alpha}}\sum_1^{n-1}\frac{1}{r^\delta}\sum_1^r\frac{e^{k^\alpha}}{k}\right\}\in BV.$$

Using Lemma 1 and the hypothesis we find $L_1\in BV$. Also

$$\begin{aligned}L_2&=\frac{1}{e^{n^\alpha}}\left(a_1-\sum_1^{n-1}\Delta(ka_k)\right)\sum_1^n\frac{e^{r^\alpha}}{r}\\&=L_{21}+L_{22},\ \text{say.}\end{aligned}$$

L_{22} is obviously of bounded variation. Now

$$L_{22}=\frac{1}{e^{n^\alpha}}\sum_{1}^{n-1}k^\delta\Delta(ka_k)\frac{1}{k^\delta}\sum_{1}^{n}\frac{e^{r^\alpha}}{r}$$

which will be of bounded variation if

$$\left\{\frac{1}{e^{n^\alpha}}\sum_{1}^{n-1}\frac{1}{k^\delta}\sum_{1}^{n}\frac{e^{r^\alpha}}{r}\right\}\in BV.$$

Since $\left\{\frac{1}{n^\alpha}\sum_{1}^{n}\frac{1}{k^\delta}\right\}\in BV$, it suffices to prove that

$$\left\{\frac{n^\alpha}{e^{n^\alpha}}\sum_{1}^{n}\frac{e^{r^\alpha}}{r}\right\}\in BV.$$

But

$$\frac{n^\alpha}{e^{n^\alpha}}\sum_{a}^{n}\frac{e^{r^\alpha}}{r}=\frac{n^\alpha}{e^{n^\alpha}}\sum_{a}^{n}\frac{(e^{r^\alpha}/r^\alpha)r^{\alpha-1}(e^{r^\alpha}/r^\alpha-e^{(r-1)^\alpha}/(r-1)^\alpha)}{e^{r^\alpha}/r-e^{(r-1)^\alpha}/(r-1)^\alpha}.$$

Since

$$\left\{\frac{(e^{r^\alpha}/r^\alpha)(r^{\alpha-1}-r^{-1})}{e^{r^\alpha}/r^\alpha-e^{(r-1)^\alpha}/(r-1)^\alpha}\right\}\in BV,$$

and a is some fixed positive integer, the result follows.

Lemma 3 [1]. *If $\sum a_n$ is summable $|R, \lambda_n, k|$, $k>0$ and* (i) $\left\{\frac{\lambda_n}{\lambda_{n+1}}\right\}\in BV$ *and* (ii) $\left\{\frac{1}{\lambda_n}\sum_{1}^{n}a_k\lambda_k\right\}\in BV$, *then* $\sum|a_n|<\infty$.

Lemma 4. *Let $f(t)\sim\sum_{1}^{\infty}a_n\cos nt$. If $\int_0^\pi\log\frac{k}{t}|df(t)|<\infty$, then $\sum a_n$ is summable $|R, e^{n^\alpha}, 1|$, where $0<\alpha<1$.*

Proof. The series $\sum a_n$ is summable $|R, e^{n^\alpha}, 1|$ if the integral

$$\int_0^\infty w^{-2}\left|\sum_{e^{n^\alpha}\leqq w}e^{n^\alpha}a_n\right|dw<\infty.$$

Now $a_n=\frac{2}{\pi}\int_0^\pi\cos nt\, f(t)dt=-\frac{2}{\pi}\int_0^\pi\frac{\sin nt}{n}df(t)$. Therefore

$$\begin{aligned}
&\int_e^\infty w^{-2}dw\left|\sum_{e^{n^\alpha}\leqq w}e^{n^\alpha}\int_0^\pi\frac{\sin nt}{n}df(t)\right|\\
&=\int_e^\infty w^{-2}dw\left|\int_0^\pi\log\frac{k}{t}\,df(t)\frac{1}{\log\frac{k}{t}}\sum_{e^{n^\alpha}\leqq w}e^{n^\alpha}\frac{\sin nt}{n}\right|\\
&\leqq\int_e^\infty w^{-2}dw\int_0^\pi\log\frac{k}{t}\,|df(t)|\frac{1}{\log\frac{k}{t}}|\eta(w,t)|\\
&=\int_0^\pi\log\frac{k}{t}\,|df(t)|\frac{1}{\log\frac{k}{t}}\int_e^\infty w^{-2}|\eta(w,t)|dw,
\end{aligned}$$

where

$$\eta(w, t)=\sum_{e^{n^\alpha}\leqslant w} e^{n^\alpha}\frac{\sin nt}{n}=\begin{cases}O(w(\log w)^{-1/\alpha}t^{-1})\\ O(w(\log w)^{-1})\end{cases} [3].$$

Since $\int_0^\pi \log\frac{k}{t}\,|df(t)|<\infty$ it is sufficient to show that

$$\int_e^\infty w^{-2}|\eta(w, t)|\,dw=O\left(\log\frac{k}{t}\right) \qquad \text{uniformly in } 0<t<\pi.$$

Let $\beta=\alpha/(1-\alpha)$ and $T=\left(\frac{k}{t}\right)^\beta$ and

$$\int_e^\infty=\int_e^{e^T}+\int_{e^T}^\infty=M_1+M_2, \text{ say.}$$

Now

$$M_1=O\left(\int_e^{e^T} w^{-1}(\log w)^{-1}dw\right)=O\left(\log\frac{k}{t}\right) \quad \text{for} \quad 0<t<\pi.$$

Also

$$\begin{aligned}M_2&=O\left(\int_{e^T}^\infty w^{-1}(\log w)^{-1/\alpha}t^{-1}dw\right)\\ &=O(t^{-1}T^{1-1/\alpha})=O\left(t^{-1}\left(\frac{k}{t}\right)^{-1}\right)=O(1)\\ &=O\left(\log\frac{k}{t}\right), \qquad 0<t<\pi.\end{aligned}$$

This completes the proof of Lemma 4.

Lemma 5 [4]. *If* $g(t)\sim\sum_1^\infty b_n \sin nt$ *and*

(2.1) $$g(+0)=0,$$

(2.2) $$\int_0^\pi \log\frac{k}{t}\,|dg(t)|<\infty$$

or, equivalently $\frac{|g(t)|}{t}\in L(0, \pi)$ *and* $g(t)\log\frac{k}{t}\in BV(0, \pi)$, *then the series* $\sum b_n$ *is summable* $|R, e^{n^\alpha}, 1|$, $0<\alpha<1$.

Lemma 6 [4]. *If* $f(t)\sim\sum_1^\infty a_n \cos nt$ *and* $f(t)\in BV(0, \pi)$, *then* $\sum\frac{a_n}{\log(n+1)}$ *is summable* $|R, e^{n^\alpha}, 1|$, $0<\alpha<1$.

Lemma 7. *Let* $f(t)\sim\sum_1^\infty a_n\cos nt$ *and* $\int_0^\pi t^{-\gamma}|df(t)|<\infty$, *then* $\sum a_n$ *is summable* $|R, e^{n(\log n)^{-\delta}}, 1|$, *where* $\gamma>0$ *and* $\delta=1+1/\gamma$.

Proof. The series $\sum a_n$ is summable $|R, e^{n(\log n)^{-\delta}}, 1|$ if

$$\int_e^\infty w^{-2}dw\left|\sum_{e^{n(\log n)^{-\delta}}\leqslant w} e^{n(\log n)^{-\delta}}a_n\right|<\infty.$$

Since $a_n=-\frac{2}{\pi}\int_0^\pi\frac{\sin nt}{n}\,df(t)$, the above integral is

$$\leqq \int_e^\infty w^{-2}dw \left| \int_0^\pi df(t) \sum_{e^{n(\log n)^{-\delta}} \leqslant w} e^{n(\log n)^{-\delta}} \frac{\sin nt}{n} \right|$$

$$\leqq \int_0^\pi t^{-r}|df(t)|\, t^r \int_e^\infty w^{-2}|g(w,t)|\,dw,$$

where

$$g(w,t) = \sum_{e^{n(\log n)^{-\delta}}} e^{n(\log n)^{-\delta}} \frac{\sin nt}{n}$$

$$= \begin{cases} O(w(\log w)^{-1}(\log\log w)^{-\delta}t^{-1}) \\ O(w(\log w)^{-1}). \end{cases}$$

By hypothesis $\int_0^\pi t^{-r}|df(t)| = O(1)$, it is therefore sufficient to prove that

$$t^r \int_e^\infty w^{-2}|g(w,t)|\,dw = O(1) \text{ uniformly in } 0<t<\pi.$$

We write

$$T = e^{e^{t^{-r}}} \text{ and } t^r \int_e^\infty = t^r \int_e^T + t^r \int_T^\infty = N_1 + N_2, \text{ say.}$$

Now

$$N_1 = O(t^r t^{-r}) = O(1) \text{ for } 0<t<\pi,$$

$$N_2 = O\left(t^{r-1} \int_T^\infty w^{-1}(\log w)^{-1}(\log\log w)^{-\delta}dw \right)$$

$$= O(t^{r-1}[(\log\log w)^{-\delta+1}]_T^\infty)$$

$$= (t^{r-1}(\log\log T)^{-1/r}) = O(t^{-1}(t^{-r})^{-1/r})$$

$$= O(t^r) = O(1) \text{ uniformly in } 0<t<\pi.$$

This completes the proof of Lemma 7.

Lemma 8. *If the sequence* $\{(\log n)^\alpha \Delta(na_n)\} \in BV$, *then*

$$\left\{ \frac{1}{e^{n(\log n)^{-\delta}}} \sum_2^n e^{k(\log k)^{-\delta}} a_k \right\} \in BV \text{ for } \alpha \geqq \delta > 0.$$

Proof. It is sufficient to assume that $\alpha = \delta$. Now

$$\frac{1}{e^{n(\log n)^{-\delta}}} \sum_2^n \frac{e^{k(\log k)^{-\delta}}}{k} k a_k$$

$$= \frac{1}{e^{n(\log n)^{-\delta}}} \sum_2^{n-1} (\log k)^\delta \Delta(k a_k) \frac{1}{(\log k)^\delta} \sum_{r=2}^k \frac{e^{r(\log r)^{-\delta}}}{r}$$

$$+ \frac{n a_n}{e^{n(\log n)^{-\delta}}} \sum_{k=2}^n \frac{e^{k(\log k)^{-\delta}}}{k} = S_1 + S_2, \text{ say.}$$

Since $\{(\log k)^\delta \Delta(k a_k)\} \in BV$, $S_1 \in BV$ by virtue of Lemma 1, provided

$$\left\{ \frac{1}{e^{n(\log n)^{-\delta}}} \sum_2^{n-1} \frac{1}{(\log k)^\delta} \sum_{r=2}^k \frac{e^{r(\log r)^{-\delta}}}{r} \right\} \in BV.$$

The above expression is of bounded variation if

$$\left\{ \frac{1}{e^{n(\log n)^{-\delta}}} \sum_2^n \frac{e^{r(\log r)^{-\delta}}}{(\log r)^\delta} \right\} \in BV.$$

This expression can be written as

$$\frac{1}{e^{n(\log n)^{-\delta}}}\sum_{a}^{n} e^{k(\log k)^{-\delta}}(\log k)^{-\delta}\frac{e^{k(\log k)^{-\delta}}-e^{(k-1)(\log (k-1))^{-\delta}}}{e^{k(\log k)^{-\delta}}-e^{(k-1)(\log (k-1))^{-\delta}}}$$

which is of bounded variation by virtue of the fact that

$$\left\{\frac{(\log k)^{-\delta}e^{k(\log k)^{-\delta}}}{e^{k(\log k)^{-\delta}}-e^{(k-1)(\log (k-1))^{-\delta}}}\right\}\in BV.$$

This completes the proof of the lemma.

3. Proof of the theorems. By virtue of Lemma 4, $\sum a_n$ is summable $|R, e^{n^\alpha}, 1|$, $0<\alpha<1$. Applying Lemma 3 the proof of Theorem 1 follows immediately. Similarly the proofs of Theorems 2, 3, and 4 are evident in view of Lemma 3 and Lemmas 5, 6, and 7 respectively.

Remark. It may be observed that the second condition in each of our theorems is lighter than the corresponding conditions of Izumi. Also these conditions are necessary for the absolute convergence of the corresponding series.

References

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