213. On Extensions of Mappings into n-Cubes

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1. Introduction. The purpose of this note is to give a generalization of the results of M. K. Fort, Jr. [1] to the case of arbitrary metric spaces.

Let X be a metric space and dim X the covering dimension of X. We denote by I^n the closed unit cube in Euclidean n-space, where $n>0$. If A is a subset of X and f is a mapping whose domain contains A, *f is of type k on A* if and only if $\dim(f^{-1}(y)\cap A)\leqq k$ for all y in the range of f, where $k\geqq -1$. In the following, a mapping means always a continuous transformation.

Let us assume that A is a closed subset of X, $\dim(X-A)=m\geqq n$ and f is a mapping of A into I^n. It will be shown that f can be extended to a mapping φ of X into I^n such that φ is of type $m-n$ on $X-A$. Under the assumption of separability for X, this theorem was proved by A. L. Gropen [2] and essentially by M. K. Fort, Jr. [1]. If f is, in addition, of type $m-n$ on A, it will also be shown that the mapping φ, whose existence is asserted above, is of type $m-n$ on X. These results will be established in §3.

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2. Auxiliary lemmas. We employ the terminology of M. K. Fort, Jr. [1]. A finite collection Σ of subsets of a metric space has Property D if and only if there exists $\varepsilon>0$ such that any set which contains at least one point from each member of Σ has diameter greater then ε. If A is a closed subset of a metric space X and f is a mapping into I^n whose domain contains A, we let $C_n(f|A)$ be the space of mappings g of X into I^n for which $g|A=f|A$ metrized by the uniform metric. By the Tietze extension theorem, $C_n(f|A)$ is nonempty and is a complete metric space.

The following Lemma 1 was proved by M. K. Fort, Jr. In his paper [1], it was assumed that X is a separable metric space, but by virtue of [5, p. 49] the separability of X is not necessary.

Lemma 1. *If A is a closed subset of a metric space X, f is a mapping of A into I^n and $F_0, \cdots, F_n$ are mutually exclusive subsets of $X-A$ which are closed in X and each of dimension less than n, then* *the set G of all mappings* $g \in C_n(f|A)$ *for which* $g(F_0), \cdots, g(F_n)$ *has Property* D *is open and dense in* $C_n(f|A)$.

Lemma 2. *Let F be a subset of a metric space X and let* $\mathfrak{U}=\{U_\gamma|\gamma \in \Gamma\}$ *be a discrete collection of subsets of X such that* $\dim(\mathfrak{B}(U_\gamma)\cap F)<k$ *for* $\gamma \in \Gamma$, *where* $\mathfrak{B}(U_\gamma)$ *denotes the boundary of* U_γ *and* $k \geqq 0$. *Then* $\dim(\mathfrak{B}(\bigcup_{\gamma\in\Gamma} U_\gamma)\cap F)<k$.

Proof. Since $\mathfrak{U}$ is a locally finite collection, we have Lemma 2 by the sum theorem [3, Theorem 3.2].

Lemma 3. *If f is a mapping of a closed subset A of a metric space X into* I^n, *F is a subset of* $X-A$ *which is closed in X and* $\dim F=m\geqq n$, *m finite, then there exists a second category set* $E\subset C_n(f|A)$ *such that each* $\varphi \in E$ *is of type* $m-n$ *on F*.

Proof. We let n be a fixed positive integer and give a proof by induction on $m-n$.

Suppose $m=n$. By A. H. Stone's theorem [6], there exists an open basis $\mathfrak{U}=\bigcup_{i=1}^{\infty}\mathfrak{U}_i$ of X, where $\mathfrak{U}_i=\bigcup_{j=1}^{\infty}\mathfrak{U}_{ij}$ is a locally finite open covering of X and $\mathfrak{U}_{ij}=\{U(i, j;\gamma)|\gamma \in \Gamma_{ij}\}$ is a discrete collection, $i, j=1, 2, \cdots$. We put $\mathfrak{U}_i=\{U(i;\gamma)|\gamma\in\Gamma_i\}$ and we may assume that

$$\text{mesh }\mathfrak{U}_i=\sup\{\text{diameter of } U(i;\gamma)|\gamma\in\Gamma_i\}<1/i.$$

Since $\mathfrak{U}_i$ is a locally finite open covering, there exists an open covering $\mathfrak{U}_i^1=\{U(i;\gamma, 1)|\gamma\in\Gamma_i\}$ of X such that $\overline{U(i;\gamma,1)}\subset U(i;\gamma)$ for $\gamma\in\Gamma_i$ and $i=1, 2, \cdots$. Continuing this process, we have locally finite open coverings $\mathfrak{U}_i^k=\{U(i;\gamma, k)|\gamma\in\Gamma_i\}$, $k=1, \cdots, n+1$, of X such that

1) $\overline{U(i;\gamma, n+1)}\subset U(i;\gamma, n)\subset\overline{U(i;\gamma, n)}\subset \cdots \subset U(i;\gamma, 1)$
$\subset\overline{U(i;\gamma,1)}\subset U(i;\gamma, 0), \qquad \gamma\in\Gamma_i,\ i=1, 2, \cdots$

where we set $U(i;\gamma, 0)=U(i;\gamma)$. By K. Morita's theorem [4, Theorem 9.1], there is a system of open sets $V(i;\gamma, k)$, $k=0, 1, \cdots, n$, $\gamma\in\Gamma_i$, $i=1, 2, \cdots$ such that

2) $\overline{U(i;\gamma, k+1)}\subset V(i;\gamma, k)\subset U(i;\gamma, k)$,

3) $\dim(\mathfrak{B}(V(i;\gamma, k))\cap F)\leqq n-1$

for $k=0, 1, \cdots, n$, $\gamma\in\Gamma_i$, $i=1, 2, \cdots$. Thus, by 1), 2), and 3), for each $U(i, j;\gamma)\in\mathfrak{U}_{ij}\subset\mathfrak{U}_i$ we have $n+1$ open sets $V(i, j;\gamma, k)$, $k=0, 1, \cdots, n$ satisfying 4) and 5):

4) $\dim(\mathfrak{B}(V(i, j;\gamma, k))\cap F)\leqq n-1$,

5) $\overline{V(i, j;\gamma, n)}\subset V(i, j;\gamma, n-1)\subset\overline{V(i, j;\gamma, n-1)}\subset \cdots$
$\subset V(i, j;\gamma, 1)\subset\overline{V(i, j;\gamma, 1)}\subset V(i, j;\gamma, 0)$

for $k=0, 1, \cdots, n$, $\gamma\in\Gamma_i$, $i, j=1, 2, \cdots$. We put

$$V(i, j; k)=\cup\{V(i, j;\gamma, k)|\gamma\in\Gamma_{ij}\}.$$

Then by Lemma 2 combined with 4) and by 5), we have

6) $\dim(\mathfrak{B}(V(i, j; k))\cap F)\leqq n-1$,

7) $\overline{V(i, j; n)} \subset V(i, j; n-1) \subset \overline{V(i, j; n-1)} \subset \cdots \subset V(i, j; 1) \subset \overline{V(i, j; 1)} \subset V(i, j; 0)$

for $k=0, 1, \cdots, n, i, j=1, 2, \cdots$. Let

$$F_{ijk} = \mathfrak{B}(V(i, j; k)) \cap F.$$

Then, by 6) and 7), $F_{ij0}, \cdots, F_{ijn}$ satisfy the conditions of Lemma 1, and therefore there exists an open dense subset $G_{ij} \subset C_n(f|A)$ such that if $g \in G_{ij}$ then the collection $g(F_{ij0}), \cdots, g(F_{ijn})$ has Property D. We let

$$E = \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{\infty} G_{ij}.$$

E is clearly a second category subset of $C_n(f|A)$.

Now let $\varphi \in E$ and $y \in I^n$. Suppose that $x \in \varphi^{-1}(y) \cap F$ and that W is a neighbourhood of x. There is an integer i such that the $1/i$-neighbourhood of x in $\varphi^{-1}(y) \cap F$ is contained in W. Therefore, since $\mathfrak{B}_i^n = \{V(i; \gamma, n) | \gamma \in \Gamma_i\}$ is an open covering of X, there exist open sets $V(i, j; \gamma, 0), \cdots, V(i, j; \gamma, n)$ such that $x \in V(i, j; \gamma, n)$ and $V(i, j; \gamma, 0) \cap \varphi^{-1}(y) \cap F \subset W$. On the other hand, since $\bigcap_{k=0}^{n} \varphi(F_{ijk}) = \emptyset$, there exists k such that $F_{ijk} \cap \varphi^{-1}(y) = \emptyset$. Since $\mathfrak{B}(V(i, j; \gamma, k)) \cap F \subset F_{ijk}$, we have

$$\mathfrak{B}(V(i, j; \gamma, k)) \cap \varphi^{-1}(y) \cap F = \emptyset.$$

Thus there exists a σ-locally finite open basis

$$\mathfrak{B} = \{V(i, j; \gamma, k) | \mathfrak{B}(V(i, j; \gamma, k)) \cap \varphi^{-1}(y) \cap F = \emptyset, \; k=0, 1, \cdots, n, \gamma \in \Gamma_{ij}, i, j=1, 2, \cdots\}$$

of $\varphi^{-1}(y) \cap F$ and $\dim(\varphi^{-1}(y) \cap F) \leqq 0$ (cf. [4, Theorem 8.7] or [5, Theorem 2.9]).

Let us assume that the lemma is true for $m-n \leqq l$, and show that this implies the lemma for $m-n=l+1$. We assume $\dim F = n+l+1$. There exists an open basis $\mathfrak{U} = \bigcup_{i=1}^{\infty} \mathfrak{U}_i$ of X, where $\mathfrak{U}_i = \bigcup_{j=1}^{\infty} \mathfrak{U}_{ij}$ is a locally finite open covering of X and $\mathfrak{U}_{ij} = \{U(i, j; \gamma) | \gamma \in \Gamma_{ij}\}$ is a discrete collection, $i, j=1, 2, \cdots$. By K. Morita's theorem, we can find an open covering $\mathfrak{B}_i = \bigcup_{j=1}^{\infty} \mathfrak{B}_{ij}$ of X, where $\mathfrak{B}_{ij} = \{V(i, j; \gamma) | \gamma \in \Gamma_{ij}\}$ is a discrete collection such that

8) $\overline{V(i, j; \gamma)} \subset U(i, j; \gamma)$,

9) $\dim(\mathfrak{B}(V(i, j; \gamma)) \cap F) \leqq n+l$

for $\gamma \in \Gamma_{ij}, i, j=1, 2, \cdots$. Let

$$V_{ij} = \cup \{V(i, j; \gamma) | \gamma \in \Gamma_{ij}\}.$$

By Lemma 2, $\dim(\mathfrak{B}(V_{ij}) \cap F) \leqq n+l, i, j=1, 2, \cdots$. By the induction hypothesis, for each i and j there exists a second category set $E_{ij} \subset C_n(f|A)$ such that if $g \in E_{ij}$ then g is of type l on $\mathfrak{B}(V_{ij}) \cap F$. We define

$$E=\bigcap_{i=1}^{\infty}\bigcap_{j=1}^{\infty}E_{ij}.$$

Clearly, $E\subset C_n(f|A)$ is a second category subset.

Let $\varphi\in E$ and $y\in I^n$. Suppose that x is a point of $\varphi^{-1}(y)\cap F$ and W is a neighbourhood of x. There exists an open set $V(i, j; \gamma)\in\bigcup_{i=1}^{\infty}\mathfrak{B}_i$ such that $x\in V(i, j; \gamma)\cap\varphi^{-1}(y)\cap F\subset W$. Since $\mathfrak{B}(V(i, j; \gamma))\subset\mathfrak{B}(V_{ij})$, we have $\dim(\mathfrak{B}(V(i, j; \gamma))\cap\varphi^{-1}(y)\cap F)\leqq l$. Therefore, $\dim(\varphi^{-1}(y)\cap F)\leqq l+1$ and φ is of type $l+1$ on F. Thus the proof of Lemma 3 is completed.

3. Extension theorem.

Theorem. *Let X be a metric space, A a closed subset of X and f a mapping from A into I^n. If $\dim(X-A)=m\geqq n$, $m<\infty$, then there exists a second category set $E\subset C_n(f|A)$ such that φ is of type $m-n$ on $X-A$ for each $\varphi\in E$.*

Proof. Since A is closed, $X-A=\bigcup_{i=1}^{\infty}F_i$, where F_i is a closed subset of X, $i=1, 2, \cdots$. Clearly, $\dim F_i\leqq m$. By Lemma 3, there exists a second category set $E_i\subset C_n(f|A)$ such that each $g\in E_i$ is of type $m-n$ on F_i, $i=1, 2, \cdots$. Letting $E=\bigcap_{i=1}^{\infty}E_i$, we have the desired set E. Indeed, by the sum theorem we have $\dim(\varphi^{-1}(y)\cap(X-A))\leqq m-n$ for $\varphi\in E$ and $y\in I^n$.

The following corollaries are easily proved by the method used in [1]. Therefore, we omit their proofs.

Corollary 1. *Let X be a metric space, A a closed subset of X and f a mapping from A into I^n. If $\dim(X-A)=m\geqq n$, $m<\infty$, then there exists a continuous extension φ of f over X such that φ is of type $m-n$ on $X-A$.*

Corollary 2. *Let X be a metric space, A a closed subset of X and f a mapping from A into I^n. If $\dim(X-A)=m\geqq n$, $m<\infty$, and f is of type $m-n$ on A, then f can be extended to a mapping φ of X into I^n such that φ is of type $m-n$ on X.*

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