212. A Note on Traces on von Neumann Algebras

By Yoshinori HAGA
Fuculty of Engineering, Ibaraki University

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The purpose of this note is to show a theorem concerning traces on von Neumann algebras, motivated by a theorem of Kakutani [4] on divergent integrals. Our theorem may be seen as an extension of Kakutani's theorem to the non-commutative abstract integral theory.

Let M^+ be the set of all positive elements of a von Neumann algebra M. A *trace* on M^+ is a functional φ defined on M^+, with values $\geqq 0$, finite or infinite, having the following properties:

(i) If $S, T \in M^+$, $\varphi(S+T)=\varphi(S)+\varphi(T)$.

(ii) If $S \in M^+$ and λ is a number $\geqq 0$, $\varphi(\lambda S)=\lambda\varphi(S)$ (here we define $0\cdot(+\infty)=0$).

(iii) If $S \in M^+$ and U is unitary, $\varphi(USU^{-1})=\varphi(S)$.

We say φ is *finite* if $\varphi(S)<+\infty$ for all $S \in M^+$, and φ is *normal* if $\varphi(\sup S_i)=\sup \varphi(S_i)$ for every uniformly bounded increasing directed set (S_i) in M^+.

Theorem. *Let M be a von Neumann algebra, and φ and ψ be normal traces on M^+. Suppose that*

$$(1) \qquad \psi(S)<+\infty \quad \textit{implies} \quad \varphi(S)<+\infty.$$

Then, there exist a positive constant K and a finite normal trace τ on M^+ such that

$$(2) \qquad \varphi(S)\leqq K\psi(S)+\tau(S) \quad \textit{for any} \quad S \in M^+.$$

This theorem concerns essentially with semi-finite von Neumann algebras because we assume the existence of normal traces, but we state and prove it without any restrictions of the types of M.

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1. Preliminary results. M^P and M^U denote the sets of all projections and unitary operators of a von Neumann algebra M respectively. Let $E, F \in M^P$. If there is a partially isometric $V \in M$ such that $V^*V=E$ and $VV^*=F$, we say E and F are *equivalent* and denote by $E \sim F$. If there is a projection F_1 such that $E \sim F_1 \leqq F$, we write $E \prec F$. Let $(E_i)_{i\in I}$ (resp. $(F_i)_{i\in I}$) be a family of mutually orthogonal projections in M, and let $E=\sum_{i\in I} E_i$ (resp. $F=\sum_{i\in I} F_i$), then E and F are also projections in M. Moreover, if $E_i \sim F_i$ (resp. $E_i \prec F_i$) for all $i \in I$, we have $E\sim F$ (resp. $E\prec F$). It is well-known that the relation $\sim$ is a usual equivalence relation, and the relations $\sim$ and $\prec$ give an order in M ([3] Chap. III. § 1.1). Next, let Z be the center of M. For any projection E in M, the minimal projection $F\in Z$ such as $F\geqq E$ is called the *central envelope* of E and we denote it by $Z(E)$. It is known that $Z(E)=\sup\{F\in M^P \mid F\sim E\}$ ([1] Lemme 3.1).

The following lemma is well-known (for example, [3] Chap. III. § 1. Theorème 1).

Lemma 1. *For any $E, F\in M^P$, there exist $E', F'\in M^P$ such that*

(i) $E'\leqq E$, $F'\leqq F$, $E'\sim F'$,

(ii) $Z(E-E')\cap Z(F-F')=0$.

The condition (ii) means that $E-E'$ and $F-F'$ have no comparable non-zero subprojections.

Using this lemma, we show the next one, which may probably be known, but we prove it for convenience sake.

Lemma 2. *Let $(E_i)_{i\in I}$ be a family of mutually orthogonal projections in M, and $F\in M$ be a projection such that $F\prec\sum_{i\in I}E_i$. Then F can be written as a sum of $(F_i)_{i\in I}$ such that $F_i\prec E_i$ for all $i\in I$.*

Proof. We well-order the set of indices I. Denoting the first index by 1, we apply Lemma 1 to E_1 and F. Then we get $E_1', F_1\in M^P$ such that

$$E_1'\leqq E_1,\quad F_1\leqq F,\quad E_1'\sim F_1$$

and

$$Z(F-F_1)\cap Z(E_1-E_1')=0.$$

Next, suppose that, for every $i<i_0$, we get an F_i such that

$$F_i\leqq F-\sum_{k<i}F_k,\qquad F_i\sim E_i'\leqq E_i$$

and

$$Z(F-\sum_{k\leqq i}F_k)\cap Z(E_i-E_i')=0.$$

Applying Lemma 1 to $F-\sum_{k<i_0}F_k$ and E_{i_0}, we get F_{i_0} such that

$$F_{i_0}\leqq F-\sum_{k<i_0}F_k,\qquad F_{i_0}\sim E_{i_0}'\leqq E_{i_0}$$

and

$$Z(F-\sum_{k\leqq i_0}F_k)\cap Z(E_{i_0}-E_{i_0}')=0.$$

Hence by transfinite induction, we get a family $(F_i)_{i\in I}$ of orthogonal subprojections of F such that $F_i\prec E_i$. If we put $G=F-\sum_{i\in I}F_i$, then

$$(3)\qquad \begin{cases} F=\sum_{i\in I}F_i+G, \\ \sum_{i\in I}E_i=\sum_{i\in I}E_i'+\sum_{i\in I}(E_i-E_i') \end{cases}$$

and

$$\sum_{i\in I}F_i\sim\sum_{i\in I}E_i'.$$

Moreover, as $Z(G)\cap Z(E_i-E_i')=0$,

$$Z(G)\cap Z(\sum_{i\in I}(E_i-E_i'))=Z(G)\cap\sup_{i\in I}(Z(E_i-E_i'))$$
$$=\sup_{i\in I}\{Z(G)\cap Z(E_i-E_i')\}=0.$$

Hence the right sides of (3) are incompatible unless $G=0$. Therefore, the assumption $F\prec\sum_{i\in I}E_i$ implies $G=0$. Q.E.D.

Next, we state about weight-functions introduced by J. von Neumann ([5] Definition 7), and about the relation between weight-functions and traces investigated by J. Dixmier [2].

A *weight-function* on M^P is a functional μ defined on M^P, having the following properties:

(i) $0\leqq\mu(E)<+\infty$ for any $E\in M^P$.

(ii) If $E_1, E_2\in M^P$ are orthogonal, $\mu(E_1+E_2)=\mu(E_1)+\mu(E_2)$.

(iii) If $E\in M^P$ and $U\in M^U$, $\mu(UEU^{-1})=\mu(E)$.

We say μ is *normal* if $\mu(\sum_{i\in I}E_i)=\sum_{i\in I}\mu(E_i)$ for any family $(E_i)_{i\in I}$ of mutually orthogonal projections ([2] Definition 6.1).

An ideal of a von Neumann algebra M is called *restricted*, if it coincides with the ideal generated by the projections it contains ([2] Definition 3.3). M itself is clearly a restricted ideal of M. Therefore, if we simply replace the restricted ideal in Proposition 10, of [2] with M, we get the following lemma.

Lemma 3. *There exists a one-to-one correspondence $\varphi\to\mu$ between finite traces on M^+ and weight-functions on M^P. This correspondence is defined by $\varphi(E)=\mu(E)$ for $E\in M^P$. φ is normal if and only if μ is normal.*

2. Proof of Theorem.

(I) First we shall show the existence of a constant K such that, for any real number $\alpha\geqq1$,

(4) $\psi(E)\leqq\alpha$ implies $\varphi(E)\leqq\alpha K$ for any $E\in M^P$.

In fact, otherwise there would exist a sequence (E_n) of projections in M such that

(5) $\psi(E_n)\leqq\alpha$ and $(\alpha+1)^n\leqq\varphi(E_n)<+\infty$.

If we put $S=\sum_{n=1}^{\infty}\frac{1}{(\alpha+1)^n}E_n$, we have

$$\|S\|\leqq\sum_{n=1}^{\infty}\frac{1}{(\alpha+1)^n}\|E_n\|=\frac{1}{\alpha}<+\infty.$$

Therefore S is an element of M^+ as a uniform limit of finite linear combinations of E_n with positive coefficients. Then, by the normality and inequalities (5)

$$\psi(S)=\sum_{n=1}^{\infty}\frac{1}{(\alpha+1)^n}\psi(E_n)\leqq\sum_{n=1}^{\infty}\frac{\alpha}{(\alpha+1)^n}<+\infty,$$

while

$$\varphi(S)=\sum_{n=1}^{\infty}\frac{1}{(\alpha+1)^n}\varphi(E_n)\geqq\sum_{n=1}^{\infty}1=+\infty.$$

This contradicts to the assumption (1). Thus we see that there exists a constant K_α depending on α such that

(6) $\quad \psi(E)\leqq\alpha$ implies $\varphi(E)\leqq\alpha K_\alpha$ for any $E\in M^P$.

We must show that these K_α can be chosen independently on α. If $(K_\alpha)_{\alpha\geqq 1}$ is bounded, we may put $K=\sup_{\alpha\geqq 1}K_\alpha$. Hence if there is no constant K independent on α, $(K_\alpha)_{\alpha\geqq 1}$ would be unbounded. Therefore, for every integer n, there would exist $\alpha_n\geqq 1$ and $E_n\in M^P$ such that

(7) $\quad \psi(E_n)\leqq\alpha_n$ and $\alpha_n n\leqq\varphi(E_n)$.

If we put $T=\sum_{n=1}^{\infty}\frac{1}{n^2\alpha_n}E_n$, we have

$$\|T\|\leqq\sum_{n=1}^{\infty}\frac{1}{n^2\alpha_n}\|E_n\|\leqq\sum_{n=1}^{\infty}\frac{1}{n^2}<+\infty.$$

Hence $T\in M^+$. Then, by the normality and (7)

$$\psi(T)=\sum_{n=1}^{\infty}\frac{1}{n^2\alpha_n}\psi(E_n)\leqq\sum_{n=1}^{\infty}\frac{1}{n^2}<+\infty,$$

while

$$\varphi(T)=\sum_{n=1}^{\infty}\frac{1}{n^2\alpha_n}\varphi(E_n)\geqq\sum_{n=1}^{\infty}\frac{1}{n}=+\infty$$

contrary to (1). This assures the existence of K.

(II) For any $E\in M^P$ with $\psi(E)<+\infty$, let α be a real number such that $0\leqq\alpha-1\leqq\psi(E)<\alpha$. Then, by the result of (I), we have $\varphi(E)\leqq\alpha K\leqq(\psi(E)+1)K$. Hence

(8) $\quad \psi(E)<+\infty$ implies $\varphi(E)-K\psi(E)\leqq K$.

Now, for any $E\in M^P$, we define

(9) $\quad \mu(E)=\sup\{\varphi(F)-K\psi(F)\,|\,F\prec E,\ \psi(F)<+\infty\}$

or equivalently

$$=\sup\{\varphi(F)-K\psi(F)\,|\,F\leqq E,\ \psi(F)<+\infty\},$$

and we shall show that μ is a normal weight-function on M^P.

(i) $0\leqq\mu(E)\leqq K$: From (8) clearly $\mu(E)\leqq K$. Put $F=0$ in (9), then $\varphi(F)-K\psi(F)=0$. Hence $\mu(E)\geqq 0$.

(ii) If $E_1, E_2\in M^P$ are orthogonal, $\mu(E_1+E_2)=\mu(E_1)+\mu(E_2)$:

$$\begin{aligned}\mu(E_1)+\mu(E_2)&=\sup\{\varphi(F_1)-K\psi(F_1)\,|\,F_1\leqq E_1,\ \psi(F_1)<+\infty\}\\&\quad+\sup\{\varphi(F_2)-K\psi(F_2)\,|\,F_2\leqq E_2,\ \psi(F_2)<+\infty\}\\&=\sup\{\varphi(F_1+F_2)-K\psi(F_1+F_2)\,|\,F_k\leqq E_k,\ \psi(F_k)<+\infty\ (k=1,2)\}\\&\leqq\sup\{\varphi(F)-K\psi(F)\,|\,F\leqq E_1+E_2,\ \psi(F)<+\infty\}\\&=\mu(E_1+E_2).\end{aligned}$$

On the other hand, making use of Lemma 2,

$$\begin{aligned}\mu(E_1+E_2)&=\sup\{\varphi(F)-K\psi(F)\,|\,F\prec E_1+E_2,\ \psi(F)<+\infty\}\\&\leqq\sup\{\varphi(F_1+F_2)-K\psi(F_1+F_2)\,|\,F_k\prec E_k,\ \psi(F_k)<+\infty\ (k=1,2)\}\\&=\sup\{\varphi(F_1)-K\psi(F_1)\,|\,F_1\prec E_1,\ \psi(F_1)<+\infty\}\end{aligned}$$

$$+\sup\{\varphi(F_2)-K\psi(F_2)\,|\,F_2\prec E_2,\ \psi(F_2)<+\infty\}$$
$$=\mu(E_1)+\mu(E_2).$$

(iii) $\mu(UEU^{-1})=\mu(E)$ $(U\in M^U)$: Obvious from $UEU^{-1}\sim E$.

(iv) *Normality*: Let $(E_i)_{i\in I}$ be a family of mutually orthogonal projections in M. Let J be any finite subset of I, then $\sum_{i\in J}E_i$ is an increasing directed set under the order defined by the inclusion of subsets J, and $\sum_{i\in I}E_i=\sup_J\sum_{i\in J}E_i$. Therefore, by the finite additivity of μ shown in (ii), we have

$$\sum_{i\in I}\mu(E_i)=\sup_J\sum_{i\in J}\mu(E_i)=\sup_J\mu(\sum_{i\in J}E_i)\leqq\mu(\sup_J\sum_{i\in J}E_i)=\mu(\sum_{i\in I}E_i).$$

On the other hand, making use of Lemma 2,

$$\mu(\sum_{i\in I}E_i)=\sup\{\varphi(F)-K\psi(F)\,|\,F\prec\sum_{i\in I}F_i,\ \psi(F)<+\infty\}$$
$$\leqq\sup\{\varphi(\sum_{i\in I}F_i)-K\psi(\sum_{i\in I}F_i)\,|\,F_i\prec E_i,\ \psi(F_i)<+\infty(i\in I)\}$$
$$=\sum_{i\in I}\sup\{\varphi(F_i)-K\psi(F_i)\,|\,F_i\prec E_i,\ \psi(F_i)<+\infty\}$$
$$=\sum_{i\in I}\mu(E_i).$$

(III) Applying Lemma 3, we extend the normal weight-function μ to a finite normal trace τ on M^+. Then, for any $E\in M^P$,

$$\varphi(E)\leqq K\psi(E)+\tau(E)$$

by (8) and (9) if $\psi(E)<+\infty$, and in the trivial sense if $\psi(E)=+\infty$. Therefore,

$$\varphi(S_n)\leqq K\psi(S_n)+\tau(S_n),$$

for operators of the form $S_n=\sum_{k=1}^{n}\lambda_kE_k$, where $(E_k)_{1\leqq k\leqq n}$ are orthogonal projections and $(\lambda_k)_{1\leqq k\leqq n}$ are positive numbers. Finally, since any $S\in M^+$ can be written as a uniform limit from below of such S_n, and φ, ψ, and τ are all normal, we can conclude

$$\varphi(S)\leqq K\psi(S)+\tau(S)\quad\text{for any}\quad S\in M^+.$$

3. In this last section, we show that our theorem includes a theorem of [4] as a special case.

Consider the measure space consisting of the unit interval $\Omega=\{\omega\,|\,0\leqq\omega\leqq1\}$, Borel sets, and Lebesgue measure. Let M be the von Neumann algebra of all multiplications by bounded measurable functions, acting on the Hilbert space $L^2(\Omega)$. Let $x(\omega)$ and $y(\omega)$ be real-valued non-negative measurable functions defined on Ω, not necessarily integrable. If we define

$$\varphi(S)=\int_\Omega x(\omega)S(\omega)d\omega$$

and

$$\psi(S)=\int_\Omega y(\omega)S(\omega)d\omega\qquad\text{for}\quad S(\omega)\in M^+,$$

we get normal traces φ and ψ on M^+, corresponding to the functions

$x(\omega)$ and $y(\omega)$ respectively. Suppose that

$$\int_{\Omega} y(\omega)S(\omega)d\omega < +\infty \quad \text{implies} \quad \int_{\Omega} x(\omega)S(\omega)d\omega < +\infty \quad \text{for any } S(\omega)\in M^+. \tag{10}$$

Then, our theorem shows that there exist a constant K and a finite normal trace τ on M^+ such that

$$\varphi(S) \leqq K\psi(S) + \tau(S) \quad \text{for any} \quad S \in M^+.$$

Since τ is normal, $\tau(S)$ can be written as follows with some non-negative and integrable function $z(\omega)$ on Ω:

$$\tau(S) = \int_{\Omega} z(\omega)S(\omega)d\omega.$$

Therefore

$$\int_{\Omega} x(\omega)S(\omega)d\omega \leqq K\int_{\Omega} y(\omega)S(\omega)d\omega + \int_{\Omega} z(\omega)S(\omega)d\omega \quad \text{for any } S(\omega)\in M^+, \tag{11}$$

and hence

$$x(\omega) \leqq Ky(\omega) + z(\omega) \quad \text{a.e.} \tag{12}$$

Thus we get the following corollary.

Corollary. *Let $x(\omega)$ and $y(\omega)$ be real-valued non-negative measurable functions on Ω, not necessarily integrable on Ω. If* (10) *is satisfied, there exist a constant K and an integrable function $z(\omega)$ which satisfy* (12).

In Theorem 1 of [4], the same conclusion was obtained under the following condition:

$$\int_{E} y(\omega)d\omega < +\infty \quad \text{implies} \quad \int_{E} x(\omega)d\omega < +\infty \quad \text{for any measurable subset } E \text{ of } \Omega. \tag{13}$$

But, in this case, the conclusion (12) implies (11), and hence (10) is also valid. Therefore (10) and (13) are equivalent. Thus, the above corollary is merely another version of Theorem 1 of [4].

References

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