211. Generalizations of the Stone-Weierstrass Approximation Theorem*)

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The celebrated Stone-Weierstrass theorem for the continuous functions on compact Hausdorff spaces has been extended to those on more general spaces [1], [3], [4], [8]. The purpose of the present note is to present some generalizations of the theorem and the Stone-Tietze extension theorem to the vector-valued continuous functions on completely regular spaces.

Let X be a completely regular space, $C(X, K)$ the algebra of all complex continuous functions (bounded or unbounded) on X and $\mathfrak{M}(C(X, K))$ the maximal ideal space of $C(X, K)$. We recall two results proved in [10], [11]: (1) $\mathfrak{M}(C(X, K))$ endowed with Stone topology (hull-kernel) is homeomorphic to the Stone-Čech compactification βX and (2) each $f \in C(X, R)$ can be extended to a continuous function $\tilde{f}$ over βX with values in $[-\infty, \infty]$. The set of all $\tilde{f}$ for $f \in C(X, K)$ is denoted by $\tilde{C}(X, K)$.

Definition 1. Let X be a completely regular space and S a subset of $C(X, K)$. A function $f \in C(X, K)$ is said to be a limit point of S under uniform topology if f can be uniformly approximated by the functions in S on subsets of X on which f is bounded.

Lemma 1. *Let X be a completely regular space and $C(X, R)$ the algebra of all real continuous functions on X. If a subalgebra S of $C(X, R)$ contains the identity element and separates $\mathfrak{M}(C(X, R))$, then S is dense in $C(X, R)$ under uniform topology. The same result holds for $C(X, R)$ if S is selfadjoint.*

Proof. By the classical Weierstrass theorem ([9], p. 175) there exists a polynomial $P_n(t)$ such that $||t|-P_n(t)|<1/n$ for $t \in [-n, n]$. Then $||f(x)|-P_n(f(x))|<1/n$ if $|f(x)| \leq n$ and $f \in S$ implies $|f| \in \bar{S}$, the closure of S. $\bar{S}$ is therefore a lattice and all $f_m=(f \wedge m) \vee (-m)$ for positive integers m and $f \in \bar{S}$ belongs to $\bar{S}$. It follows that the bounded functions in $\bar{S}$ separates the compact Hausdorff space $\mathfrak{M}(C(X, R))$ and all bounded real continuous functions on X are elements of $\bar{S}$ as a consequence of the Stone-Weierstrass theorem. Since

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every unbounded continuous functions on X is a limit point of $C^*(X, R)$, we have $\bar{S}=C(X, R)$.

Lemma 2. *Let T be a compact Hausdorff space and $\tilde{C}(T, R)$ a closed algebra of continuous real functions on T under uniform topology with values in $[-\infty, \infty]$ separating T, and with the property that each $\tilde{f} \in \tilde{C}(X, R)$ is finitely valued on a dense subset X of T. If $\tilde{S}$ is subalgebra of $\tilde{C}(X, R)$ which separates T and contains constant functions, then the closure of $\tilde{S}$ under uniform topology is $\tilde{C}(T, R)$.*

The same proof for Lemma 1 can be applied and it is easy to see that T is homeomorphic to $\mathfrak{M}(C(X, R))$.

Lemma 3. *Let X be a completely regular space and S a subset of $C(X, R)$. The sets of constancy for $\tilde{S}$ in βX constitute an upper semicontinuous decomposition of βX* ([7], p. 126).

Proof. Let E be a closed set in βX. Denote by E' the union of all the sets of constancy which intersect E and let x_0 be a limit point of E'. For any finite set $\pi=\{f_1, \cdots, f_p; g_1, \cdots, g_q; h_1, \cdots, h_r\} \subset S$, define $H_n(\pi)=\{x: |f_i(x)-f_i(x_0)| \leq 1/n,\ g_j(x) \geq n,\ h_k(x) \leq -n,\ i=1, \cdots \cdots, p,\ j=1, \cdots, q,\ k=1, \cdots, r\}$ if $g_j(x_0)=\infty$, $h_k(x_0)=-\infty$ and $f_i(x_0)$ are finite. As x_0 is a limit point of E', $E \cap H_n(\pi)$ is nonempty. The compactness of E and the finite intersection property of the set of sets $H_n(\pi)$ imply that all $H_n(\pi)$ have a common point $x_1 \in E$. Then x_0 and x_1 belong to the same set of constancy and $x_0 \in E'$. The upper semicontinuity of the decomposition of βX is proved.

Lemma 4. *Let X be a completely regular space and S_0 a selfadjoint subalgebra of $C(X, K)$ which contains constant functions and is contained in a closed subalgebra S of $C(X, K)$ under uniform topology. If $f \in C(X, K)$ and $\tilde{f} \in \tilde{S}$ on every set of constancy for $\tilde{S}_0$ in $\mathfrak{M}(C(X, K))$, then f belongs to S.*

Proof. Assume that S_0 is closed. The set $\sum$ of sets of constancy for $\tilde{S}_0$ in βX constitute a compact Hausdorff space if any subset Ω of $\sum$ is defined as open when the union of the sets in Ω is open in βX. For each $\xi_0 \in \sum$ there is $\tilde{g}_0 \in \tilde{S}$ with $\tilde{f}(x)=\tilde{g}_0(x)$ for $x \in \xi_0$. Let $V=\{x: |\tilde{f}(x)-\tilde{g}_0(x)|<\varepsilon,\ x \in \beta X$, and $g_0(x)$ for $x \in \xi_0$ is finite; $f(x)>\frac{1}{\varepsilon}$ or $f(x)<-\frac{1}{\varepsilon}$ otherwise$\}$. Then V is an open set containing ξ_0. Since $\sum$ is an upper semicontinuous decomposition of βX by Lemma 3, all the sets of constancy contained in V form an open set W in $\sum$. Let $\{W_1, \cdots, W_n\}$ be an open covering of the compact Hausdorff space $\sum$ and $\tilde{g}_1, \cdots, \tilde{g}_n$ the corresponding functions. Denote the union of the sets of constancy contained in V_i by U_i. Then $|\tilde{f}(x)-\tilde{g}_i(x)|<\varepsilon$, for $x \in U_i$ and $|f(x)| \leq \frac{1}{\varepsilon}$. The partition of unity $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$ on the compact space $\sum$ subordinated to the open covering $\{W_1, \cdots, W_n\}$ corresponds to the continuous functions $\tilde{u}_1, \cdots, \tilde{u}_n$ on βX. By Lemma 2, $\tilde{u}_1, \cdots, \tilde{u}_n$ belong to $\tilde{S}_0$ and thus $\tilde{g}=\tilde{u}_1\tilde{g}_1+\cdots+\tilde{u}_n\tilde{g}_n \in \tilde{S}$. Since $|\tilde{f}(x)-\tilde{g}(x)| \leq \sum_{i=1}^{n} |\tilde{u}_i(x)| \, |\tilde{f}(x)-\tilde{g}_i(x)| < \varepsilon$ and $|f(x)| \leq \frac{1}{\varepsilon}$, for $x \in \beta X$ then f belongs to S and the proof is complete.

Lemma 4 generalizes the Silov-Stone-Weierstrass theorem ([7], p. 126).

Definition 2. Let A be a complete commutative seminormed $*$-algebra with a family $\mathfrak{B}$ of seminorms and with the identity element. A is called regular if, for each closed maximal ideal M_0 of A, there is $x_0 \in M_0$ such that $\bar{V}=\sup\{V : V(x_0) \leqq 1, \ V \in \mathfrak{B}\}$ is a seminorm in $\mathfrak{B}$.

We have proved in [11] that a regular complete commutative seminormed $*$-algebra with identity is isometric (seminorm preserving) and $*$-isomorphic to $C(T, K)$, where T is a locally compact Hausdorff space.

Lemma 5. *Let X be a completely regular space and $C(X, A)$ the algebra of all continuous functions defined on X with values in a regular complete commutative seminormed $*$-algebra A with identity. The space $\mathfrak{M}(C(X, A))$ of all maximal ideals in $C(X, A)$ topologized in the Stone's sense is homeomorphic to the Stone-Čech compactification $\beta\{\mathfrak{M}_1(A) \times X\}$ of the product space $\mathfrak{M}_1(A) \times X$, $\mathfrak{M}_1(A)$ being the space of all closed maximal ideals in A* [11].

Proof. A is algebraically $*$-isomorphic and topologically isometric to the algebra $C(T, K)$, equipped with compact-open topology, of all continuous complex functions on a locally compact Hausdorff space T which is equivalent to $\mathfrak{M}_1(A)$ [11]. There exists an isometric $*$-isomorphism between $C(X, A)$ and $C(T \times X)$, i.e., between $C(X, A)$ and $C(\mathfrak{M}_1(A) \times X)$. Then $\mathfrak{M}(C(X, A))$ endowed with Stone topology is homeomorphic to $\mathfrak{M}[C(\mathfrak{M}_1(A) \times X, K)]$ or $\beta\{\mathfrak{M}_1(A) \times X\}$ [10].

Lemma 5 is an analogue to a theorem due to Yood and Hausner [5].

Definition 3. Let X be a completely regular space and A a complete regular commutative seminormed $*$-algebra. To each $f \in C(X, A)$ there corresponds a unique $f_1 \in \tilde{C}[\beta\{\mathfrak{M}_1(A) \times X\}, K]$. Define $f \vee g$ ($f \wedge g$) for $f, g \in C(X, A)$ as the element corresponding to $f_1 \vee g_1$ ($f_1 \wedge g_1$) for corresponding $f_1, g_1 \in \tilde{C}[\beta\{\mathfrak{M}_1(A) \times X\}, K]$. Also define a function $f \in C(X, A)$ as a uniform limit of a subalgebra S of $C(X, A)$ if $\tilde{f}_1$ is a limit of the corresponding subalgebra $\tilde{S}$ of $\tilde{C}[\beta\{\mathfrak{M}_1(A) \times X\}, K]$ under uniform topology.

Theorem 1. *Let X be a completely regular space and A a regular complete commutative seminormed $*$-algebra with identity. If*

$S_0(X, A)$ is a selfadjoint subalgebra of $C(X, A)$ which contains vector-valued constant functions and is contained in closed subalgebra $S(X, A)$ of $C(X, A)$, then $f \in C(X, A)$ and $\tilde{f} \in \tilde{S}$ on every set of constancy for $\tilde{S}_0(X, \tilde{A})$ in $\beta\{\mathfrak{M}_1(A) \times X\}$ imply that f belongs to $S(X, A)$. ($\tilde{A}$ is the union of A and $\{\pm\infty \cdot e\}$).

The theorem is an immediate consequence of Lemma 4, Lemma 5, and Definition 3.

Corollary. *If a $*$-subalgebra $S(X, A)$ of $C(X, A)$ contains vector-valued constant functions and separates $\mathfrak{M}\{C(X, A)\}$, then $S(X, A)$ is dense in $C(X, A)$ under uniform topology.*

Lemma 6. *Let X be a completely regular space, E a compact set in X and $E_0 \subset E$ a set dence in E. Let $\tilde{C}(E_0, R)$ be the algebra of all real continuous functions on E with values in $[-\infty, \infty]$ and assuming finite values on E_0. If $\tilde{G}_0(E_0, R)$ is any subset of $\tilde{C}(E_0, R)$ and $\mathfrak{C}(\tilde{G}_0)$ the family of all functions generated from $\tilde{G}_0$ by the lattice operations and completed under uniform topology, then a necessary and sufficient condition for a function $\tilde{f} \in \tilde{C}(E_0, R)$ to be in $\mathfrak{C}(\tilde{G}_0)$ is that, for any positive integer n, any $\varepsilon > 0$ and any two points $x, y \in E_n = \{x : |f(x)| \leq n, x \in E\}$, there exist a function $f_{xy} \in \mathfrak{C}(\tilde{G}_0)$ such that $|f(x) - f_{xy}(x)| < \varepsilon$, $|f(y) - f_{xy}(y)| < \varepsilon$.*

The lemma is an analogue of a theorem due to Stone ([9], p. 170) and can be derived by applying Stone's theorem for the functions on compact sets E_n for positive integers n. The following is consequence of Lemma 6 by identifying X and E as the same compact Hausdorff space $\mathfrak{M}(C(E_0, R))$.

Lemma 7. *If X is a completely regular space, $S(X, R)$ a closed linear sublattice of $C(X, R)$ under uniform topology containing constant functions and if there exists for any two distinct points $M_1, M_2 \in \mathfrak{M}(C(X, R))$ a continuous functions $f \in C(X, R)$ such that $\tilde{f}(M_1) \neq \tilde{f}(M_2)$, then $S(X, R) = C(X, R)$ (see [6], Theorem 4).*

Theorem 2. *Let X, A be as before and let E_0 be any subset of X. Every function belonging to $C(X_0, A)$ has a continuous extension over βX if and only if $C(X, A)$ separates $\mathfrak{M}(C(X_0, A))$.*

Proof. It suffices to show that every function $\in C(\mathfrak{M}_1(A) \times E_0, K)$ has a continuous extension over $\beta\{\mathfrak{M}_1(A) \times X\}$ if $\tilde{C}(\mathfrak{M}_1(A) \times X, K)$ separates $\mathfrak{M}[C(\mathfrak{M}_1(A) \times E_0, K)]$. $\mathfrak{M}[C(\mathfrak{M}_1(A) \times E_0, K)]$ is a compact subset of $\beta\{\mathfrak{M}_1(A) \times X\}$ and E_0 is dense in $\mathfrak{M}[C(\mathfrak{M}_1(A) \times E_0, K)]$. Following Stone's idea ([9], p. 242) and using Lemma 7, we see that the set of functions generated from the restrictions of $\tilde{C}[\beta\{\mathfrak{M}_1(A) \times X\}, K]$ on $\mathfrak{M}[\tilde{C}(\mathfrak{M}_1(A) \times E_0, K)]$ by the lattice operations and completed under uniform topology is, in fact, $\tilde{C}(\mathfrak{M}_1(A) \times E_0, K)$.

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