210. *Semifield Valued Functionals on Linear Spaces*

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An ordered (real) linear space E is defined as a linear space with an order relation satisfying the following conditions:

1) $x \leqslant y$ implies $x+z \leqslant y+z$,

2) $x \leqslant y$, $0 \leqslant \alpha$ imply $\alpha x \leqslant \alpha y$

for every x, y in E. Then $K=\{x \mid x \geqslant 0\}$ is a convex cone, i.e., K has the following properties:

3) $K+K \subset K$,

4) $\alpha K \subset K$ for every positive real number α,

5) $K \cap (-K)=$the zero element of E.

As well known, for a real linear space, there is a one-to-one correspondence between all order relations 1), 2) and all convex sets with properties 3)–5). For details of ordered linear spaces, [1], [3]–[5].

In this Note, we shall consider semifield valued functionals on E. Unless the contrary is mentioned, functionals mean semifield valued functionals.

We shall prove a theorem which is a generalization of our result [2]. In our discussion, we follow the techniques by M. Cotlar and R. Cignoli [1].

Theorem 1. *Let E be an ordered linear space, K its associated cone, and G a linear subspace of E. Let $p(x)$ be a sublinear functional on E, $f(x)$ a linear functional on F satisfying*

$$(1) \qquad f(y) \ll p(y+z) \quad \textit{for all} \quad y \in G,\ z \in K.$$

Then there is a linear extension $F(x)$ on E of $f(x)$ such that

$$F(x) \ll p(x+z) \quad \textit{for all} \quad x \in E,\ z \in K.$$

The notion of semifields was introduced by M. Antonovski, V. Boltjanski and T. Sarymsakov. For the notations used, see my reviews of their books, Zentralblatt für Mathematik, **142**, pp. 209–211 (1968).

Proof. Let $E-G \neq \emptyset$, and we take an element $x_0 \in E-G$. Then each element x of the linear space $G_1=(G, x_0)$ generated by G and x_0 is uniquely represented in the form of $x=x' \pm \alpha x_0 (x' \in G, \alpha > 0)$. We shall extend the linear functional $f(x)$ on G_1. Consequently, by the transfinite method or the Zorn lemma, we have a linear functional $F(x)$ satisfying the conditions mentioned in Theorem 1.

Let $y_1, y_2 \in G$, $z_1, z_2 \in K$, then by (1) we have

$$f(y_1)+f(y_2)=f(y_1+y_2)\ll p(y_1+y_2+z_1+z_2)$$
$$\ll p(y_1+x_0+z_1+y_2-x_0+z_2)$$
$$\ll p(y_1+x_0+z_1)+p(y_2-x_0+z_2).$$

Hence

$$-p(y_2-x_0+z_2)+f(y_2)\ll p(y_1+x_0+z_1)-f(y_1).$$

Since $y_1, y_2 \in G$, $z_1, z_2 \in K$ are arbitrary, we have

$$a=\sup_{\substack{y_2\in G\\ z_2\in K}}\{-p(y_2-x_0+z_2)+f(y_2)\}\ll$$
$$b=\inf_{\substack{y_1\in G\\ z_1\in K}}\{p(y_1+x_0+z_1)-f(y_1)\}.$$

Take an elemant c such that $a\ll c\ll b$. Then, for $y_1\in G$, $z_1\in K$,

(2) $$f(y_1)+c\ll p(y_1+x_0+z_1),$$

and, for $y_2\in G$, $z_2\in K$,

(3) $$f(y_2)-c\ll p(y_2-x_0+z_2).$$

Define f^* by

$$f^*(x'\pm\alpha x_0)=f(x')\pm\alpha c,\quad x'\in G,\quad \alpha>0.$$

Then it is obvious that f^* is a linear functional on G_1. By (2), (3), we have

$$f(\alpha y_1)+\alpha c\ll p(\alpha y_1+\alpha x_0+\alpha z_1),$$
$$f(\alpha y_2)-\alpha c\ll p(\alpha y_2-\alpha x_0+\alpha z_2).$$

Therefore those inequalities imply

$$f(x')\pm\alpha c\ll p(x'+\alpha x_0+z),$$

which is

$$f^*(x)\ll p(x+z)\quad\text{for}\quad x\in G_1,\ z\in K.$$

Therefore the proof of Theorem 1 is complete.

Theorem 2. *Let E be an ordered linear space, K its associated cone. Let $f(x)$ be a linear functional defined on a linear subspace G of E, $p(x)$ a sublinear functional on E. Then the following conditions are equivalent:*

1) *$f(x)\ll p(x+z)$ for $x\in G$, $z\in K$,*

2) *There is a linear extension $F(x)$ on E of $f(x)$ such that $0\ll F(x)$ on K and $F(x)\ll p(x)$ on E.*

Proof. 1)⇒2). By Theorem 1, there is a linear functional $F(x)$ on E such that $F(x)\ll p(x+z)$ for $x\in E$, $z\in K$. Hence $F(x)\ll p(x)$. Put $x=-z$ in $F(x)\ll p(x+z)$, then we have

$$F(-z)\ll p(-z+z)=0\quad\text{for}\quad z\in K.$$

Hence $-F(z)\ll 0$ on K, which is $0\ll F(z)$ on K.

2)⇒1). $0\ll F(z)$ on K implies

$$f(x)=F(x)\ll F(x)+F(z)=F(x+z)\ll p(x+z)$$

for $x\in G$, $z\in K$. Therefore the proof of Theorem 2 is complete.

Theorem 3. *Let E be an ordered linear space, K its associated cone. Let $f(x)$ be a linear functional on E, $p(x)$ a sublinear functional,*

$g(x)$ a linear functional on a linear subspace F of E. Then the following conditions are equivalent:

1) *There is a linear extension $G(x)$ on E of $g(x)$ such that*

$$f(z) \ll G(z) \quad for \quad z \in K, \tag{1}$$

$$G(x) \ll p(x) \quad for \quad x \in E. \tag{2}$$

2) *There is a linear extension $G(x)$ on E of $g(x)$ such that*

$$G(x) + f(z) \ll p(x+z) \quad for \quad x \in E,\ z \in K. \tag{3}$$

3) *The functional $g(x)$ satisfies*

$$g(x) + p(z) \ll p(x+z) \quad for \quad x \in E,\ z \in K. \tag{4}$$

The formulation is due to [1].

Proof. 1)$\Rightarrow$2). Let $G(x)$ be a linear extension of $g(x)$ satisfying the conditions (1), (2). Then, for $x \in E$, $z \in K$,

$$G(x) + f(z) \ll G(x) + G(z) = G(x+z) \ll p(x+z).$$

2)$\Rightarrow$3). It is obvious that (3) implies (4).

3)$\Rightarrow$1). Let $p_1(x) = p(x) - f(x)$, $g_1(x) = g(x) - f(x)$, then by (4), we have

$$(g_1(x) + f(x)) + f(z) \ll p_1(x+z) + f(x+z).$$

Hence $g_1(x) \ll p_1(x+z)$ for $y \in F$, $z \in K$. By Theorem 1, there is a linear functional $G_1(x)$ such that $G_1(x)$ is an extension of $g_1(x)$, and $G_1(x) \ll p_1(x+z)$ for $x \in E$, $z \in K$. Therefore

$$G_1(x) \ll p_1(x+z) = p(x+z) - f(x) - f(z),$$

which implies

$$G_1(x) + f(x) + f(z) \ll p(x+z)$$

for $x \in E$, $z \in K$. Let $G(x) = G_1(x) + f(x)$, then for $x \in F$,

$$G(x) = G_1(x) + f(x) = g_1(x) + f(x) = g(x).$$

Therefore $G(x)$ is an extension of $g(x)$, and from

$$G(x) + f(z) = G_1(x) + f(x) + f(z) \ll p(x+z),$$

we have $G(x) \ll p(x)$ for $x \in E$, and $f(z) \ll G(z)$ for $z \in K$. The proof of Theorem 3 is complete.

References

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