207. *On Axiom Systems of Commutative Rings*

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Recently G. R. Blakley gives an interesting axiom system of commutative rings (see G. R. Blakley [1]).

In this short Note, we give some new axioms of commutative rings and semirings that the addition and multiplication are commutative.

Theorem 1. *A set with two nullary operations,* 0 *and* 1, *with one unary operation,* $-$, *and with two binary operations,* $+$ *and juxtaposition such that*

1) $r+0=r,$

2) $r1=r,$

3) $((-r)+r)a=0,$

4) $((ay+bx)+cz)r=b(xr)+(a(yr)+z(cr))$

for every a, b, c, r, x, y, z, *is a commutative ring with unit element.*

Remark. It is obvious that every commutative ring (with unit element) satisfies 1)—4).

Proof. The proof is divided into several steps.

5) $(-r)+r$
$=((-r)+r)1$ {2}
$=0.$ {3}

6) $0a$
$=((-0)+0)a$ {5}
$=0.$ {3}

7) $a+b$
$=((a1+b1)+00)1$ {2, 6}
$=b(11)+(a(11)+0(01))$ {4}
$=b+a.$ {2, 6, 1}

8) cz
$=((00+00)+cz)1$ {1, 7, 6, 2}
$=0(01)+(0(01)+z(c1))$ {4}
$=zc.$ {1, 7, 2}

9) $(b+a)+c$
$=(a+b)+c$ {7}
$=((a1+b1)+c1)1$ {2}
$=b(11)+(a(11)+1(c1))$ {4}
$=b+(a+c).$ {2}

10) $(ay)r$
$=((ay+00)+00)r$ {1, 6}
$=0(0r)+(a(yr)+0(0r))$ {4}
$=a(yr).$ {6, 1, 7}

11) $(a+b)r$
$=((a1+b1)+00)r$ {2, 1, 6}
$=b(1r)+(a(1r)+0(0r))$ {4}
$=br+ar$ {2, 8, 6, 1}
$=ar+br.$ {7}

12) For given a, b, $a+x=b$ is solvable. Let $x=(-a)+b$, then we have

$a+((-a)+b)$
$=(a+(-a))+b$ {9}
$=b.$ {5, 1, 7}

Therefore the proof of Theorem 1 is complete.

Theorem 2. *A set with two nullary operations,* 0 *and* 1, *with two binary operations,* + *and juxtaposition such that*

1) $r+0=r,$
2) $r1=r,$
3) $0r=0,$
4) $((ay+bx)+cz)r=b(xr)+(a(yr)+z(cr))$

for every a, b, c, r, x, y, z, *is a semiring with* 0 *and* 1 *that these binary operations satisfy the commutative laws.*

Proof. We divide our proof into some steps.

5) $a+b$
$=((a1+b1)+00)1$ {2, 1, 3}
$=b(11)+(a(11)+0(01))$ {4}
$=b+a.$ {2, 3, 1}

6) ab
$=((00+00)+ab)1$ {1, 5}
$=0(01)+(0(01)+b(a1))$ {4}
$=ba.$ {3, 1, 5}

7) $(b+a)+c$
$=(a+b)+c$ {5}
$=((a1+b1)+c1)1$ {2}
$=b(11)+(a(11)+1(c1))$ {4}
$=b+(a+c).$ {2, 6}

8) $(ay)r$
$=((ay+00)+00)r$ {1}
$=0(0r)+(a(yr)+0(0r))$ {4}
$=a(yr).$ {1, 5, 3}

9)	$(a+b)r$	
	$=((a1+b1)+00)r$	{2, 1}
	$=b(1r)+(a(1r)+0(0r))$	{4}
	$=br+ar$	{2, 3}
	$=ar+br.$	{5}

Hence the proof of Theorem 2 is complete.

Next we consider another axiom system which characterizes a commutative ring.

Theorem 3. *A set with two nullary operations,* 0 *and* 1, *with one unary operation,* $-$, *and with two binary operations,* $+$ *and juxtaposition such that*

1) $r+0=0+r=r,$
2) $r1=r,$
3) $((-r)+r)a=0,$
4) $((ay+b)+c)r=rb+(a(yr)+cr)$

for every $a, b, c, r, y,$ *is a commutative ring with unit element.*

Remark. It is obvious that every commutative ring (with unit element) satisfies 1)—4).

Proof.

5)	$(-r)+r$	
	$=((-r)+r)1$	{2}
	$=0.$	{3}
6)	$0a$	
	$=((-0)+0)a$	{5}
	$=0.$	{3}
7)	br	
	$=((00+b)+0)r$	{1, 6}
	$=rb+(0(0r)+0r)$	{4}
	$=rb.$	{6, 1}
8)	$a+b$	
	$=((a1+b)+0)1$	{2, 1}
	$=1b+(a(11)+01)$	{4}
	$=b+a.$	{2, 7, 6, 1}
9)	$(ay)r$	
	$=((ay+0)+0)r$	{1}
	$=r0+(a(yr)+0r)$	{4}
	$=a(yr).$	{6, 7, 1}
10)	$(a+b)+c$	
	$=(b+a)+c$	{8}
	$=((b1+a)+c)1$	{2}
	$=1a+(b(11)+c1)$	{4}
	$=a+(b+c).$	{2, 7}

11) $(a+b)r$

$=((a1+b)+0)r$ {2, 1}

$=rb+(a(1r)+01)$ {4}

$=rb+ar$ {2, 6, 1}

$=ar+br.$ {8}

12) For given a, b, $a+x=b$ is solvable. Let $x=(-a)+b$, then we have

$a+((-a)+b)$

$=(a+(-a))+b$ {10}

$=b.$ {5, 8, 1}

Therefore the proof of Theorem 3 is complete.

Theorem 4. *A set with two nullary operations,* 0 *and* 1, *with two binary operations,* + *and juxtaposition, such that*

1) $r+0=0+r=r,$

2) $r1=r,$

3) $0r=0,$

4) $((ay+b)+c)r=rb+(a(yr)+cr)$

for every a, b, c, r, y, *is a commutative semiring.*

Remark. It is obvious that every commutative semiring (with unity) satisfies 1)—4).

Proof.

5) br

$=((00+b)+0)r$ {1, 3}

$=rb+(0(0r)+0r)$ {4}

$=rb.$ {3, 1}

6) $a+b$

$=((a1+b)+0)1$ {2, 1}

$=1b+(a(11)+01)$ {4}

$=b+a.$ {2, 5, 3, 1}

7) $(ay)r$

$=((ay+0)+0)r$ {1}

$=r0+(a(yr)+0r)$ {4}

$=a(yr).$ {3, 5, 1}

8) $(a+b)+c$

$=(b+a)+c$ {6}

$=((b1+a)+c)1$ {2}

$=1a+(b(11)+c1)$ {4}

$=a+(b+c).$ {2, 5}

9) $(a+b)r$

$=((a1+b)+0)r$ {2, 1}

$=rb+(a(1r)+0r)$ {4}

$=rb+ar$ {2, 3, 1}

$$=ar+br. \qquad \{5, 6\}$$

Hence the proof of Theorem 4 is complete.

Reference

[1] G. R. Blakley: Four axioms for commutative rings. Notices of Amer. Math. Soc., **15**, p. 730 (1968).