201. On Numbers Expressible as a Weighted Sum of Powers

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(Comm. by Zyoiti SUETUNA, M. J. A., Nov. 12, 1968)

1. In a recent paper [3] we proved

Theorem 1. *There is n_0 such that for every $n \geqq n_0$ there are positive integers x and y satisfying*

$$n < x^f + y^h < n + cn^p$$

where f and h are any integers such that $f \geqq h \geqq 2$,

$$c = hf^{1-(1/h)} \text{ and } p = \left(1-\frac{1}{f}\right)\left(1-\frac{1}{h}\right).$$

Mordell [4] has recently proved

Theorem 2. *There are non-negative integers $x_1, \cdots, x_k$ satisfying*

$$n \leqq a_1 x_1^{h_1} + \cdots + a_k x_k^{h_k} < n + cn^p + O(n^{p(h_k-2)/(h_k-1)})$$

where $a_1, \cdots, a_k \geqq 1$, $1 < h_1 \leqq h_2 \leqq \cdots \leqq h_k$,

$$c = (a_1^{1/h_1} h_1)(a_2^{1/h_2} h_2)^{1-(1/h_1)}(a_3^{1/h_3} h_3)^{(1-(1/h_1))(1-(1/h_2))} \cdots (a_k^{1/h_k} h_k)^{(1-(1/h_1))\cdots(1-(1/h_{k-1}))}$$

and

$$p = \left(1-\frac{1}{h_1}\right) \cdots \left(1-\frac{1}{h_k}\right).$$

Theorem 1 generalizes some results previously obtained by Bambah and Chowla [1], Uchiyama [5] and the author [2] while Theorem 2 deals with a problem more general than those discussed in [1], [5], [2] and [3].

In this note we prove the following generalization of Theorem 1 and refinement of Theorem 2:

Theorem 3. *There is n_0 such that for every real $n \geqq n_0$ there are positive integers $x_1, \cdots, x_k$ satisfying*

$$n < a_1 x_1^{h_1} + \cdots + a_k x_k^{h_k} < n + cn^p$$

where $a_1, \cdots, a_k$ are real and >0, $h_1, \cdots, h_k$ are real and >1, $k>1$, c and p are as in Theorem 2 and

$$a_1 h_1^{h_1} \leqq a_2 h_2^{h_2} \leqq \cdots \leqq a_k h_k^{h_k}.$$

In what follows we write $[t]$ for the greatest integer $\leqq t$.

2. We first prove the following generalization of Theorem 4A of [2]:

Theorem 4. *Let a and $b>0$, f and $h>1$,*

$$N = N(n) = a\{(n/a)^{1/f}+1\}^f - n + b$$

and

$$g(n)=N-b\{(N/b)^{1/h}-1\}^h.$$

Then for every $n\geqq a$ *there are positive integers* x *and* y *satisfying*

$$n<ax^f+by^h<n+g(n).$$

Proof. Clearly N increases with n and $g(n)$ with N. Thus $N>b$ and $g(n)>b$. Thus the theorem is clearly true if $(n/a)^{1/f}$ is an integer. In the rest of the proof we therefore assume that

(1) $$m=[(n/a)^{1/f}]<(n/a)^{1/f}.$$

The theorem is clearly true if

$$a(m+1)^f+b<n+g(n).$$

In the rest of the proof we therefore assume that

(2) $$a(m+1)^f+b\geqq n+g(n).$$

Since $m=[(n/a)^{1/f}]\geqq 1$ and

$$am^f+b\left[\left(\frac{n-am^f}{b}\right)^{1/h}+1\right]^h\leqq am^f+b\left\{\left(\frac{n-am^f}{b}\right)^{1/h}+1\right\}^h$$

the theorem follows easily from

Lemma 1. (1) *and* (2) *imply that*

$$am^f+b\left\{\left(\frac{n-am^f}{b}\right)^{1/h}+1\right\}^h<n+g(n).$$

Proof. From (2)

$$n-am^f\leqq a(m+1)^f-am^f+b-g(n).$$

Clearly $(m+1)^f-m^f$ increases with m. Hence from (1)

$$\begin{aligned}n-am^f&<a\{(n/a)^{1/f}+1\}^f-n+b-g(n)\\&=N-g(n)=b\{(N/b)^{1/h}-1\}^h.\end{aligned}$$

Hence

$$\begin{aligned}am^f+b\left\{\left(\frac{n-am^f}{b}\right)^{1/h}+1\right\}^h&=n+b\left\{\left(\frac{n-am^f}{b}\right)^{1/h}+1\right\}^h-(n-am^f)\\&<n+N-b\{(N/b)^{1/h}-1\}^h=n+g(n)\end{aligned}$$

since $b\left\{\left(\frac{n-am^f}{b}\right)^{1/h}+1\right\}^h-(n-am^f)$ clearly increases with $n-am^f$. This completes the proof.

We next prove the following generalization of Theorem 1:

Theorem 5. *There is* n_0 *such that for every* $n\geqq n_0$ *there are positive integers* x *and* y *satisfying*

$$n<ax^f+by^h<n+b^{1/h}h(a^{1/f}f)^{1-(1/h)}n^{(1-(1/f))(1-(1/h))}$$

where a *and* $b>0$ *and* f *and* $h>1$.

Proof. We have

$$\begin{aligned}N(n)&=a\{(n/a)^{1/f}+1\}^f-n+b\\&=f(n/a)^{1-(1/f)}\{1+O(n^{-1/f})+O(n^{(1/f)-1})\}\end{aligned}$$

as $n\to\infty$. Hence

$$\begin{aligned}g(n)&=N-b\{(N/b)^{1/h}-1\}^h\\&=b^{1/h}hN^{1-(1/h)}\left\{1-\frac{h-1}{2}\left(\frac{N}{b}\right)^{-1/h}+O(N^{-2/h})\right\}\end{aligned}$$

$$=b^{1/h}h(a^{1/f}f)^{1-(1/h)}n^{(1-(1/f))(1-(1/h))}$$
$$\cdot\left\{1-\frac{h-1}{2}\left(\frac{N}{b}\right)^{-1/h}+O(N^{-2/h})+O(N^{-1/(f-1)})+O(N^{-1})\right\}$$

as $n\to\infty$. Now $-2/h$, -1 and $-1/(f-1)<-1/h$ if $f-1<h$. Hence $g(n)<b^{1/h}h(a^{1/f}f)^{1-(1/h)}n^{(1-(1/f))(1-(1/h))}$ for large n. Thus Theorem 5 follows from Theorem 4 and

Lemma 2. *Theorem 5 is true if $f\geqq h+1$.*

Proof. Let q be a fixed constant such that

$$\frac{1}{2}f<q<f.$$

Suppose first that

(3) $$m=[(n/a)^{1/f}]\geqq\{(n/a)-q(n/a)^{1-(1/f)}\}^{1/f}.$$

Then

$$n<am^f+b\left[\left(\frac{n-am^f}{b}\right)^{1/h}+1\right]^h$$
$$\leqq am^f+b\left\{\left(\frac{n-am^f}{b}\right)^{1/h}+1\right\}^h$$
$$\leqq n+b^{1/h}h\{qa(n/a)^{1-(1/f)}\}^{1-(1/h)}\{1+o(1)\}$$
$$<n+b^{1/h}h(a^{1/f}f)^{1-(1/h)}n^{(1-(1/f))(1-(1/h))}$$

for large n, since $q<f$. Hence the lemma follows if (3) be true. We therefore assume in the rest of the proof that (3) is false, i.e., that

(4) $$m=[(n/a)^{1/f}]<\{(n/a)-q(n/a)^{1-(1/f)}\}^{1/f}=M, \text{ say.}$$

Lemma 3. *Let $f\geqq h+1$,*

$$P=a(M+1)^f-aM^f+b$$

and

$$Q(n)=P-b\{(P/b)^{1/h}-1\}^h.$$

Then $Q(n)<b^{1/h}h(a^{1/f}f)^{1-(1/h)}n^{(1-(1/f))(1-(1/h))}$ for large n.

Proof. For large n,

$$M=(n/a)^{1/f}\{1-(q/f)(n/a)^{-1/f}+o(n^{-1/f})\},$$
$$P=afM^{f-1}\left\{1+\frac{1}{2}(f-1)M^{-1}+o(M^{-1})\right\}$$
$$=a^{1/f}fn^{1-(1/f)}\left\{1-q\left(1-\frac{1}{f}\right)\left(\frac{n}{a}\right)^{-1/f}+o(n^{-1/f})\right\}$$
$$\cdot\left\{1+\frac{1}{2}(f-1)\left(\frac{n}{a}\right)^{-1/f}+o(n^{-1/f})\right\}$$
$$=a^{1/f}fn^{1-(1/f)}\left\{1-\left(q-\frac{f}{2}\right)\left(1-\frac{1}{f}\right)\left(\frac{n}{a}\right)^{-1/f}+o(n^{-1/f})\right\}$$

and

$$Q(n)=b^{1/h}hP^{(h-1)/h}\left\{1-\frac{1}{2}(h-1)\left(\frac{P}{b}\right)^{-1/h}+o(P^{-1/h})\right\}$$
$$<b^{1/h}h(a^{1/f}f)^{1-(1/h)}n^{(1-(1/f))(1-(1/h))},$$

since $q>(1/2)f$.

Suppose now that

$$a(m+1)^f+b\leqq n+Q(n).$$

Then Lemma 2 is clearly true. We therefore assume in the rest of the proof that

$$(5)\qquad a(m+1)^f+b>n+Q(n).$$

Since

$$n<am^f+b\left[\left(\frac{n-am^f}{b}\right)^{1/h}+1\right]^h$$
$$\leqq am^f+b\left\{\left(\frac{n-am^f}{b}\right)^{1/h}+1\right\}^h,$$

Lemma 2 now follows from Lemma 3 and

Lemma 4. (4) *and* (5) *imply that*

$$am^f+b\left\{\left(\frac{n-am^f}{b}\right)^{1/h}+1\right\}^h<n+Q(n).$$

Proof. From (5),

$$n-am^f<a(m+1)^f-am^f+b-Q(n).$$

Clearly $(m+1)^f-m^f$ increases with m. Hence, from (4),

$$n-am^f<a(M+1)^f-aM^f+b-Q(n)$$
$$=P-Q(n)=b\{(P/b)^{1/h}-1\}^h.$$

Hence

$$am^f+b\left\{\left(\frac{n-am^f}{b}\right)^{1/h}+1\right\}^h=n+b\left\{\left(\frac{n-am^f}{b}\right)^{1/h}+1\right\}^h-(n-am^f)$$
$$<n+P-b\{(P/b)^{1/h}-1\}^h=n+Q(n),$$

since $b\left\{\left(\frac{n-am^f}{b}\right)^{1/h}+1\right\}^h-(n-am^f)$ increases with $n-am^f$. This completes the proof.

Remark 1. It is clear that in Theorem 5 we can replace the coefficient $b^{1/h}h(a^{1/f}f)^{1-(1/h)}$ of $n^{(1-(1/f))(1-(1/h))}$ by $a^{1/f}f(b^{1/h}h)^{1-(1/f)}$, obtaining an improvement only if $af^f<bh^h$.

3. Proof of Theorem 3. The case $k=2$ follows easily from Theorem 5. For $k>2$ we assume that $n\geqq a_k$ and let $x_k=[(n/a_k)^{1/h_k}]$. If $n-a_kx_k^{h_k}\geqq a_{k-1}$ we obtain Theorem 3, by induction on k, using the fact that

$$n-a_kx_k^{h_k}\leqq n-a_k\{(n/a_k)^{1/h_k}-1\}^{h_k}<h_ka_k(n/a_k)^{1-(1/h_k)}$$

for large n. Otherwise we obtain Theorem 3 using the fact that $a_1+\cdots+a_{k-1}<cn^p$ for large n.

Remark 2. In the above proof we have not used the inequalities $a_1h_1^{h_1}\leqq\cdots\leqq a_kh_k^{h_k}$ appearing in the statement of Theorem 3. If they are not all true we can improve the result by replacing c by a smaller constant C. This is seen as follows:

Let $a_ih_i^{h_i}>a_{i+1}h_{i+1}^{h_{i+1}}$ for some i satisfying $1\leqq i\leqq k-1$. Clearly c

can be replaced by C, obtained from c by interchanging a_i with a_{i+1} and h_i with h_{i+1}. It is not difficult to see that $C<c$ since $a_i h_i^{h_i} > a_{i+1} h_{i+1}^{h_{i+1}}$.

I am indebted to Professor Mordell for communicating his result to me before the publication of his paper.

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