234. On the Mixed Problem for the Wave Equation with an Oblique Derivative Boundary Condition

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1. Introduction. On the mixed problem for hyperbolic equations, only few results have been derived. Indeed, up to now, even for second order equations only the problems with the Dirichlet type boundary condition and with the Neumann type boundary condition are studied satisfactorily, and concerning the wave equation we don't know whether the problem with an oblique derivative boundary condition is well posed or not.

In this note, we show that the above problem in a half space is not well posed in L^2-sense.

At first we explain the well-posedness in L^2sense. Let Ω be a sufficiently smooth domain in R^n, L be a second order hyperbolic operator with coefficients in $\mathscr{B}(\Omega\times[0,T])$ and $B=b_1\left(x,t:\dfrac{\partial}{\partial x}\right)+b_2(x,t)\dfrac{\partial}{\partial t}$ be a first order differential boundary operator. Consider the mixed problem

$$
(\mathrm{P})\quad\begin{cases}(1.1)\quad L[u(x,t)]=f(x,t) & \text{in}\quad \Omega\times(0,T)\\ (1.2)\quad Bu(x,t)=0 & \text{on}\quad \partial\Omega\times[0,T]\\ (1.3)\quad u(x,0)=u_0(x),\ \dfrac{\partial}{\partial t}(x,0)=u_1(x). & \end{cases}
$$

Definition. *The mixed problem* (P) *is said to be well posed in L^2-sense if for any initial data* $\{u_0(x),u_1(x)\}\in N=\{(u,v):u\in H^2(\Omega)$, $v\in H^1(\Omega)$ *satisfying* $b_1\left(x,0:\dfrac{\partial}{\partial x}\right)u+b_2(x,0)v=0$ *on* $\partial\Omega\}$ *there exists one and only one solution of* (P) *in* $\mathcal{E}_t^0(H^2(\Omega))\cap\mathcal{E}_t^1(H^1(\Omega))\cap\mathcal{E}_t^2(L^2(\Omega))$[1] *satisfying* $L[u]=0$ *and the following energy inequality holds for* $t\in[0,T]$.

$$
(1.4)\qquad \|u(x,t)\|_{1,L^2(\Omega)}^2+\left\|\frac{\partial u}{\partial t}(x,t)\right\|_{L^2(\Omega)}^2\leqslant C(\|u_0(x)\|_{1,L^2(\Omega)}^2+\|u_1(x)\|_{L^2(\Omega)}^2).
$$

Then our result is

Theorem. *In the case* $\Omega=\{(x,y):x>0,\ -\infty<y<\infty\}$, $L=\dfrac{\partial^2}{\partial t^2}-\dfrac{\partial^2}{\partial x^2}-\dfrac{\partial^2}{\partial y^2}$ *and* $B=\dfrac{\partial}{\partial x}+b\dfrac{\partial}{\partial y}$, *where b is any non zero real constant, then the mixed problem* (P) *is not well posed in L^2-sense.*

Hereafter we denote by (P_0) the mixed problem for the above Ω,

1) $f\in\mathcal{E}_t^k(E)$ means that f is k times continuously differentiable in t as E-valued function.

L, B.

2. Some lemmas.

Lemma 2.1. *Assume that* (P) *is well posed in* L^2*-sense and the coefficients of* L *and* B *are independent of* t. *Then for any initial data* $\{u_0(x), u_1(x)\} \in N$ *the solution* $u(x, t)$ *exists for* $t \in [0, \infty)$ *and for some constant* $\gamma > 0$ *the energy inequality*

$$\|u(x,t)\|^2_{1,L^2(\Omega)} + \left\|\frac{\partial u}{\partial t}(x,t)\right\|^2_{L^2(\Omega)} \leq Ce^{\gamma t}(\|u_0(x)\|^2_{1,L^2(\Omega)} + \|u_1(x)\|^2_{L^2(\Omega)}) \tag{2.1}$$

holds moreover the higher energy inequality

$$\|u(x,t)\|^2_{2,L^2(\Omega)} + \left\|\frac{\partial u}{\partial t}(x,t)\right\|^2_{1,L^2(\Omega)} + \left\|\frac{\partial^2 u}{\partial t^2}(x,t)\right\|^2_{L^2(\Omega)} \leq Ce^{\gamma t}(\|u_0(x)\|^2_{2,L^2(\Omega)} + \|u_1(x)\|^2_{1,L^2(\Omega)}) \tag{2.2}$$

also holds.

Lemma 2.2. *Let* p, q, k *be constants such that* $\mathrm{Im}\, p > 0$, $\mathrm{Im}\, q < 0$, *and* $p + k \neq 0$. *For any* $f(x) \in L^2(R_+)$ *the solution* $w(x)$ *in* $L^2(R_+)$ *of*

$$\begin{cases} \left(\dfrac{1}{i}\dfrac{d}{dx} - p\right)\left(\dfrac{1}{i}\dfrac{d}{dx} - q\right)w(x) = f(x) \quad x > 0 \\ \left(\dfrac{1}{i}\dfrac{d}{dx} + k\right)w(x)\Big|_{x=0} = 0 \end{cases}$$

exists uniquely and $w(0)$ *and* $\dfrac{dw}{dx}(0)$ *are given by the formulas*

$$w(0) = \frac{i}{p+k}\int_0^\infty e^{-iql} f(l)\,dl \tag{2.4}$$

$$\frac{dw}{dx}(0) = \frac{k}{p+k}\int_0^\infty e^{-iql} f(l)\,dl. \tag{2.5}$$

3. Proof of Theorem. Without loss of generality we assume that b is a positive constant. Let us prove Theorem by contradiction.

Assume that (P_0) is well posed in L^2-sense. Let $u(x, y, t)$ be the solution for $u_0 \equiv 0$, $u_1(x, y) = g(x, y)$ and $f \equiv 0$, i.e. $u(x, y, t) \in \mathcal{E}^0_t(H^2(R^2_+)) \cap \mathcal{E}^1_t(H^1(R^2_+)) \cap \mathcal{E}^2_t(L^2(R^2_+))$ satisfies

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)u(x,y,t) = 0 \ \text{ in } R^2_+ \times (0, \infty) \tag{3.1}$$

$$\left[\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}\right]_{x=0} = 0, \tag{3.2}$$

$$u(x,y,0) = 0, \ \frac{\partial u}{\partial t}(x,y,0) = g(x,y).$$

And from Lemma 2.1

$$E(t) \leqslant Ce^{\gamma t}\|g(x,y)\|^2_{L^2(R^2_+)}, \tag{3.3}$$

where

$$E(t) = \left\|\frac{\partial u}{\partial t}(x,y,t)\right\|^2_{L^2(R^2_+)} + \left\|\frac{\partial u}{\partial x}(x,y,t)\right\|^2_{L^2(R^2_+)} + \left\|\frac{\partial u}{\partial y}(x,y,t)\right\|^2_{L^2(R^2_+)}.$$

On the other hand by using the integration by parts and (3.1)

$$\frac{d}{dt}E(t)=\int_{-\infty}^{\infty}\left\{\frac{\partial u}{\partial t}(0,y,t)\overline{\frac{\partial u}{\partial x}(0,y,t)}+\frac{\partial u}{\partial x}(0,y,t)\overline{\frac{\partial u}{\partial t}(0,y,t)}\right\}\,dy.$$

Put

$$S(t)=\int_{-\infty}^{\infty}\left\{\frac{\partial u}{\partial t}(0,y,t)\overline{\frac{\partial u}{\partial x}(0,y,t)}+\frac{\partial u}{\partial x}(0,y,t)\overline{\frac{\partial u}{\partial t}(0,y,t)}\right\}\,dy$$

then

$$E(t)-E(0)=\int_0^t S(l)dl. \tag{3.4}$$

Since we have by using the higher energy inequality,

$$|S(t)|\leqslant\text{const.}\ e^{rt}\|g(x,y)\|^2_{1,L^2(R^2_+)},$$

$$\int_0^\infty e^{-2\mu t}\left(\int_0^t S(l)dl\right)dt=\frac{1}{2\mu}\int_0^\infty e^{-2\mu t}S(t)dt, \tag{3.5}$$

it follows from (3.3), (3.4) and (3.5) that

$$\left|\frac{1}{2\mu}\int_0^\infty e^{-2\mu t}S(t)dt\right|\leqslant\frac{1+C}{2(\mu-\gamma)}\|g(x,y)\|^2_{L^2(R^2_+)}\,. \tag{3.6}$$

Denote by $\hat{u}(x,\eta,\tau)$ the image of Fourier-Laplace transformation of $u(x,y,t)$, i.e.,

$$\hat{u}(x,\eta,\tau)=\int_0^\infty dt\int_{-\infty}^\infty dy\; e^{-i\eta y}e^{-\tau t}\,u(x,y,t).$$

By Parseval's theorem we get

$$\int_0^\infty e^{-2\mu t}dt\int_{-\infty}^\infty dy\frac{\partial u}{\partial x}(0,y,t)\overline{\frac{\partial u}{\partial t}(0,y,t)}$$
$$=\int_{-\infty}^\infty \partial\nu\int_{-\infty}^\infty d\eta\frac{\partial\hat{u}}{\partial x}(0,\eta,\tau)\overline{\tau\hat{u}(0,\eta,\tau)},$$

where τ denotes the complex number $\mu+i\nu$. Then from (3.6)

$$\left|\text{Re}\int_{-\infty}^\infty d\nu\int_{-\infty}^\infty d\eta\frac{\partial\hat{u}}{\partial x}(0,\eta,\tau)\overline{\tau\hat{u}(0,\eta,\tau)}\right| \tag{3.7}$$
$$\leqslant\frac{\mu(1+C)}{2(\mu-\gamma)}\|g(x,y)\|^2.$$

Since $\hat{u}(x,\eta,\tau)$ satisfies for all $\tau=\mu+i\nu$ and almost everywhere of η real the equations

$$\begin{cases}\left(\left(\dfrac{1}{i}\dfrac{d}{dx}\right)^2+\eta^2+\tau^2\right)\hat{u}(x,\eta,\tau)=\tilde{g}(x,\eta) & x>0,\\ \left(\dfrac{1}{i}\dfrac{d}{dx}+b\eta\right)\hat{u}(x,\eta,\tau)\,|_{x=0}=0,\end{cases} \tag{3.8}$$

where $\tilde{g}(x,\eta)$ is the Fourier image with respect to y of $g(x,y)$.

The application of Lemma 2.2 for (3.8) gives

$$\frac{d\hat{u}}{dx}(0,\eta,\tau)=\frac{b\eta}{i\sqrt{\eta^2+\tau^2}+b\eta}\int_0^\infty e^{-\sqrt{\eta^2+\tau^2}x}\tilde{g}(x,\eta)\,dx \tag{3.9}$$

$$\tau\hat{u}(0,\eta,\tau)=\frac{i\tau}{i\sqrt{\eta^2+\tau^2}+b\eta}\int_0^\infty e^{-\sqrt{\eta^2+\tau^2}x}\tilde{g}(x,\eta)\,dx \tag{3.10}$$

where $\sqrt{\eta^2+\tau^2}$ denotes the square root of $\eta^2+\tau^2$ with positive real part.

Let us choose a sequence of data $g_n(x, y)$ given in the form

(3.11) $$\tilde{g}_n(x, \eta) = P_n(\eta)\sqrt{n}\, h(nx).$$

Put

(3.12) $$J_n(\eta) = \int_{-\infty}^{\infty} d\nu \frac{\mathrm{Re}(b\eta\, \overline{i\tau})}{|i\sqrt{\eta^2+\tau^2}+b\eta|^2} \left| \int_0^\infty e^{-\sqrt{\eta^2+\tau^2}x}\sqrt{n}\, h(nx)dx \right|^2.$$

Lemma 3.1. *Let $h(x)$ be a real valued C^∞-function with a compact support contained in $[1, \infty)$ such that*

(3.13) $$\int_0^\infty e^{-ibx} h(x)dx \neq 0.$$

Then it holds for some constant $C_0 > 0$

(3.14) $$-J_n(n) \geqslant C_0 n.$$

Proof. At first remark that

(3.15) $$J_n(n) = -b\int_{-\infty}^{\infty} d\nu \frac{\nu}{|\sqrt{1+(\mu'+i\nu)^2}-ib|^2} \left| \int_0^\infty e^{-\sqrt{1+(\mu'+i\nu)^2}x} h(x)dx \right|^2$$

where $\mu' = \dfrac{\mu}{n}$, and

(3.16) $$\int_0^\infty e^{-\sqrt{1+(\mu'+i\nu)^2}x} h(x)dx = \overline{\int_0^\infty e^{-\sqrt{1+(\mu'-i\nu)^2}x} h(x)dx}.$$

And we note some properties of $\sqrt{1+\tau^2}$: For all $\nu > 0$

(3.17) $$\frac{1}{|\sqrt{1+(\mu'+i\nu)^2}-ib|^2} - \frac{1}{|\sqrt{1+(\mu'-i\nu)^2}-ib|^2} \geqslant 0$$

and, when $\nu \in I_{\mu'} = [\sqrt{1+b^2+\mu'^2}-\mu', \sqrt{1+b^2+\mu'^2}+\mu']$, for some $C > 0$

(3.18) $$\frac{1}{|\sqrt{1+(\mu'+i\nu)^2}-ib|^2} - \frac{1}{|\sqrt{1+(\mu'-i\nu)^2}-ib|^2} \geqslant \frac{1}{C^2}\frac{1}{\mu'^2} - \frac{1}{b^2}.$$

From (3.15) and (3.16) it follows

$$-J_n(n) = b\int_{-\infty}^{\infty} \nu\, d\nu \left\{ \frac{1}{|\sqrt{1+(\mu'+i\nu)^2}-ib|^2} - \frac{1}{|\sqrt{1+(\mu'-i\nu)^2}-ib|^2} \right\} \cdot \left| \int_0^\infty e^{-\sqrt{1+(\mu'+i\nu)^2}x} h(x)dx \right|^2,$$

by (3.17) and (3.18)

$$\geqslant b\int_{I_{\mu'}} \nu \left(\frac{1}{C^2}\frac{1}{\mu'^2} - \frac{1}{b^2} \right) \left| \int_0^\infty e^{-\sqrt{1+(\mu'+i\nu)^2}x} h(x)dx \right|^2 d\nu.$$

On the other hand since for $\nu \in I_{\mu'}$ $\sqrt{1+(\mu'+i\nu)^2}$ tends to ib when μ' tends to 0 it holds from (3.13)

$$\left| \int_0^\infty e^{-\sqrt{1+(\mu'+i\nu)^2}x} h(x)dx \right|^2 > C_1$$

for all $\nu \in I_{\mu'}$ if μ' is sufficiently small. Then it follows that

$$-J_n(n) \geqslant b\frac{\sqrt{1+b^2}}{2} \left(\frac{1}{C^2}\frac{1}{\mu'^2} - \frac{1}{b^2} \right) C_1 2\mu' \geqslant \text{const.}\ \frac{1}{\mu'} = \text{const}\frac{n}{\mu}.$$

Thus Lemma is proved.

Substituting (3.11) into (3.9) and (3.10) we see that

$$\text{(3.19)}\qquad \mathrm{Re}\int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} d\eta \frac{\partial \hat{u}}{\partial x}(0, \eta, \tau)\overline{\tau\hat{u}(0, \eta, \tau)} = \int_{-\infty}^{\infty} |P_n(\eta)|^2 J_n(\eta) d\eta.$$

Since $J_n(\eta)$ is continuous in η, by choosing $P_n(\eta)$ as its support contained in a sufficiently small neighborhood of n, it follows

$$\text{(3.20)}\qquad -\int_{-\infty}^{\infty} |P_n(\eta)|^2 J_n(\eta) d\eta \geqslant -\frac{J_n(n)}{2}\int_{-\infty}^{\infty} |P_n(\eta)|^2 d\eta.$$

By inserting (3.19) into (3.7) and using (3.20) and (3.14) it follows

$$\frac{C_0}{2} n \int_{-\infty}^{\infty} |P_n(\eta)|^2 d\eta \leqslant \frac{(1+C)\mu}{2(\mu-\gamma)} \|g_n(x, y)\|^2_{L^2(R^2_+)} = \frac{(1+C)\mu}{2(\mu-\gamma)} \int_{-\infty}^{\infty} |P_n(\eta)|^2 d\eta \int_0^{\infty} |h(x)|^2 dx,$$

this shows that

$$\frac{C_0}{2} n \leqslant \frac{(1+C)\mu}{2(\mu-\gamma)} \int_0^{\infty} |h(x)|^2 dx$$

holds for all n. This is a contradiction. Thus Theorem is proved.

Remark. By the same method we can also see the following:

Let Ω and L are the same ones in Theorem. If we take the boundary operator as

$$B = \frac{\partial}{\partial x} + c\frac{\partial}{\partial t},$$

the mixed problem

$$\begin{cases} Lu = 0 & \text{in } R^2_+ \times (0, T) \\ Bu|_{x=0} = 0 & \\ u(x, y, 0) = u_0(x, y), & \dfrac{\partial u}{\partial t}(x, y, 0) = u_1(x, y) \end{cases}$$

is not well posed in L^2-sense when c is a negative constant.

References

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