230. On Proofs of Some Axioms with Sheffer Functor 'D'

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There has been represented an extension of "Fitch's rules" for Sheffer functor by D. F. Siemens [3]. There is a set of four rules for the 'Sheffer stroke' or 'alternative denial', symbolized by '|' or 'D'. The relation between this functor and other propositional functors, and some deductions from ordinary substitution and detachment rules are shown in K. Iséki [1] and T. W. Scharle [2].

The rule for Introduction (DI):

1	| | p	H (hypothesis)
⋮	| | ⋮	
2	| | q	assumption (i.e., it is assumed that this deduction can be completed)
3	| | Dqq	assumption
4	| Dpp	1, 2, 3, DI.

The rule for Elimination (DE):

1	| Dpq	H (given)
2	| | Dpp	1, DE (4)
⋮	| | ⋮	
3	| | r	assumption
4	| | Dqq	1, DE (2)
⋮	| | ⋮	
5	| | r	assumption
6	| r	2–3, 4–5, DE.

The rule for substitution (DSF) consists of the following pair:

1	| Dpp	given	1	| Dpp	given
2	| Dpq	1, DSF	2	| Dqp	1, DSF.

The rule for Contraction (DCF):

1	| $DDppDpp$	given
2	| p	1, DCF.

The rule for reiteration to the same proof level or to an inner level, 'R', is also needed. Nicod's rule of transformation can be asserted:

1	| p	given
2	| $DpDrq$	given
3	| q	1, 2, Nicod.

For the details of the proof of this rule, see [3].

In this paper, we shall prove some rules, symbolized, '*DCF**', and further prove some axiom systems.

Let *DCF** be a simple extension of *DCF*. *DCF** has three kinds of type, that is, given '*DDpqDrp*', '*p*' can be asserted, and given '*DDpqDpq*', '*p*' can be asserted, '*q*' can be asserted. For the proof of the first type, see D. F. Siemens [3]. We shall prove other two rules.

The proof of the second type (*DCF**):

```
 1  | DDpqDpq              H (hypothesis)
    |---------
 2  |  | DDpqDpq           1, DE (7)
    |  |---------
 3  |  |  | Dpp            H
    |  |  |-----
 4  |  |  | Dpq            3, DSF
 5  |  |  | DDpqDpq        2, R (i.e., reiteration)
 6  |  | DDppDpp           3, 4, 5, DI
 7  |  | DDpqDpq           1, DE (2)
    |  |---------
 8  |  | DDppDpp           the next steps are identical to steps 3–6.
 9  | DDppDpp              2–6, 7–8, DE
10  | p                    9, DCF (1, DCF*).
```

The proof of the third type (*DCF**):

```
 1  | DDpqDpq              H
    |---------
 2  |  | DDpqDpq           1, DE (7)
    |  |---------
 3  |  |  | Dqq            H
    |  |  |-----
 4  |  |  | Dpq            3, DSF
 5  |  |  | DDpqDpq        2, R
 6  |  | DDqqDqq           3, 4, 5, DI
 7  |  | DDpqDpq           1, DE (2)
    |  |---------
 8  |  | DDqqDqq           The next steps are identical to steps 3–6.
 9  | DDqqDqq              2–6, 7–8, DE
10  | q                    9, DCF (1, DCF*).
```

Docin's rule of the transformation provided that, given '*p*' and '*DpDrq*', '*r*' can be asserted in this set of rules.

Proof.

```
 1  | p                    H (given)
 2  | DpDrq                H (given)
    |-------
 3  |  | Dpp               2, DE (10)
    |  |-----
 4  |  | p                 1, R
 5  |  |  | Drr            H
    |  |  |-----
 6  |  |  | p              4, R
 7  |  |  | Dpp            3, R
 8  |  | DDrrDrr           5, 6, 7, DI
 9  |  | r                 8, DCF
10  |  | DDrqDrq           2, DE (3)
11  |  | r                 10, DCF* (demonstrated above)
```

12 | *r* 2–9, 10–11, *DE* (1, 2, Docin).

There have been given some single axiom systems for functor '*D*':

(1) *DDpDqrDDtDttDDsqDDpsDps* (by J. Nicod),

(2) *DDpDqrDDsDssDDsqDDpsDps* (by J. Lukasiewicz),

(3) *DDpDqrDDpDrpDDsqDDpsDps* (by J. Lukasiewicz),

(4) *DDpDqrDDDsrDDpsDpsDpDpq* (by M. Wajsberg).

Nicod's axiom has been proved in D. F. Siemens' paper [3]. In this paper, we shall prove (2)–(4).

Proof of (2).

1	│ │ *DDDpDqrDDsDssDDsqDDpsDpsDDp-DqrDDsDssDDsqDDpsDps*	*H*
2	│ │ *DpDqr*	1, *DCF**
3	│ │ *DDsDssDDsqDDpsDps*	1, *DCF**
4	│ │ │ *DDsDssDsDss*	3, *DE* (11)
5	│ │ │ *s*	4, *DCF**
6	│ │ │ *Dss*	4, *DCF**
7	│ │ │ │ *Dps*	*H*
8	│ │ │ │ *s*	5, *R*
9	│ │ │ │ *Dss*	6, *R*
10	│ │ │ *DDpsDps*	7, 8, 9, *DI*
11	│ │ │ *DDDsqDDpsDpsDDsqDDpsDps*	3, *DE* (4)
12	│ │ │ *DDpsDps*	11, *DCF**
13	│ │ *DDpsDps*	4–10, 11–12, *DE*
14	│ │ *p*	13, *DCF**
15	│ │ *s*	15, *DCF**
16	│ │ │ *DDsDssDsDss*	3, *DE* (23)
17	│ │ │ *s*	16, *DCF**
18	│ │ │ *Dss*	16, *DCF**
19	│ │ │ │ *DDsqDDpsDps*	*H*
20	│ │ │ │ *s*	17, *R*
21	│ │ │ │ *Dss*	18, *R*
22	│ │ │ *DDDsqDDpsDpsDDsqDDpsDps*	19, 20, 21, *DI*
23	│ │ │ *DDDsqDDpsDpsDDsqDDspDps*	3, *DE* (16)
24	│ │ │ *DDDsqDDpsDpsDDsqDDpsDps*	23, *R*
25	│ │ *DDDsqDDpsDpsDDsqDDpsDps*	16–22, 23–24, *DE*
26	│ │ *Dsq*	25, *DCF**
27	│ │ *q*	2, 14, Docin
28	│ │ │ *Dss*	26, *DE* (30)
29	│ │ │ *Dss*	28, *R*
30	│ │ │ *Dqq*	26, *DE* (28)
31	│ │ │ *q*	27, *R*
32	│ │ │ │ *s*	*H*

33	*q*	31, *R*
34	*Dqq*	30, *R*
35	*Dss*	32, 33, 34, *DI*
36	*Dss*	28–29, 30–35, *DE*
37	*DDDDpDqrDDsDssDDsqDDpsDpsDDp-DqrDDsDssDDsqDDpsDpsDDDpDqr-DDsDssDDsqDDpsDpsDDpDqrDDsDss-DDsqDDpsDps*	15, 36, *DI*
38	*DDpDqrDDsDssDDsqDDpsDps*	37, *DCF*

Proof of (3).

1	*DDDpDqrDDpDrpDDsqDDpsDpsDDp-DqrDDpDrpDDsqDDpsDps*	*H*
2	*DpDqr*	1, *DCF**
3	*DDpDrpDDsqDDpsDps*	1, *DCF**
4	*DDpDrpDpDrp*	3, *DE* (6)
5	*p*	4, *DCF**
6	*DDDsqDDpsDpsDDsqDDpsDps*	3, *DE* (4)
7	*DDpsDps*	6, *DCF**
8	*p*	7, *DCF**
9	*p*	4–5, 6–8, *DE*
10	*q*	2, 9, Docin
11	*r*	2, 9, Nicod
12	*DDpDrpDpDrp*	3, *DE* (31)
13	*p*	9, *R*
14	*r*	11, *R*
15	*Drp*	12, *DCF**
16	*Drr*	15, *DE* (23)
17	*r*	14, *R*
18	*Dss*	*H*
19	*r*	17, *R*
20	*Drr*	16, *R*
21	*DDssDss*	18, 19, 20, *DI*
22	*s*	21, *DCF*
23	*Dpp*	15, *DE* (16)
24	*p*	13, *R*
25	*Dss*	*H*
26	*p*	24, *R*
27	*Dpp*	23, *R*
28	*DDssDss*	25, 26, 27, *DI*
29	*s*	28, *DCF*
30	*s*	16–22, 23–29, *DE*
31	*DDDsqDDpsDpsDDsqDDpsDps*	3, *DE* (12)
32	*DDpsDps*	31, *DCF**

33	s	32, DCF^*
34	s	12–30, 31, 33, DE
35	$DDpDrpDpDrp$	3, DE (49)
36	Drp	35, DCF^*
37	Drr	36, DE (42)
38	$DDsqDsq$	H
39	r	11, R
40	Drr	37, R
41	$DDDsqDsqDDsqDsq$	38, 39, 40, DI
42	Dpp	36, DE (37)
43	$DDsqDsq$	H
44	p	9, R
45	Dpp	42, R
46	$DDDsqDsqDDsqDsq$	43, 44, 45, DI
47	$DDDsqDsqDDsqDsq$	37–41, 42–45, DE
48	Dsq	47, DCF
49	$DDDsqDDpsDpsDDsqDDpsDps$	3, DE (35)
50	Dsq	49, DCF^*
51	Dsq	35–48, 49–50, DE
52	Dss (See steps 26–36 in the proof of (2).)	10, 51
53	$DDDDpDqrDDpDrpDDsqDDpsDpsDDp$-$DqrDDpDrpDDsqDDpsDpsDDDpDqr$-$DDpDrpDDsqDDpsDpsDDpDqrDDp$-$DrpDDsqDDpsDps$	1, 34, 52, DI
54	$DDpDqrDDpDrpDDsqDDpsDps$	53, DCF

Proof of (4).

1	$DDDpDqrDDDsrDDpsDpsDpDpqDDp$-$DqrDDDsrDDpsDpsDpDpq$	H
2	$DpDqr$	1, DCF^*
3	$DDDsrDDpsDpsDpDpq$	1, DCF^*
4	$DDDsrDDpsDpsDDsrDDpsDps$	3, DE (7)
5	$DDpsDps$	4, DCF^*
6	p	5, DCF^*
7	$DDpDpqDpDpq$	3, DE (4)
8	p	7, DCF^*
9	p	4–6, 7–8, DE
10	q	2, 9, Docin
11	r	2, 9, Nicod
12	$DDDsrDDpsDpsDDsrDDpsDps$	3, DE (15)
13	$DDpsDps$	12, DCF^*
14	s	13, DCF^*
15	$DDpDpqDpDpq$	3, DE (12)
16	Dpq	15, DCF^*

```
17  │ │ │ │Dpp                                    16, DE (22)
18  │ │ │ │ │Dss                                  H
19  │ │ │ │ │p                                    9, R
20  │ │ │ │ │Dpp                                  17, R
21  │ │ │ │DDssDss                                18, 19, 20, DI
22  │ │ │ │Dqq                                    16, DE (17)
23  │ │ │ │ │Dss                                  H
24  │ │ │ │ │q                                    10, R
25  │ │ │ │ │Dqq                                  22, R
26  │ │ │ │DDssDss                                23, 24, 25, DI
27  │ │ │DDssDss                                  17–21, 22–26, DE
28  │ │ │s                                        27, DCF
29  │ │s                                          12–14, 15–28, DE
30  │ │ │DDDsrDDpsDpsDDsrDDpsDps                  3, DE (32)
31  │ │ │Dsr                                      30, DCF*
32  │ │ │DDpDpqDpDpq                              3, DE (30)
33  │ │ │Dpq                                      32, DCF*
34  │ │ │ │Dpp                                    33, DE (39)
35  │ │ │ │ │DDsrDsr                              H
36  │ │ │ │ │p                                    9, R
37  │ │ │ │ │Dpp                                  34, R
38  │ │ │ │DDDsrDsrDDsrDsr                        35, 36, 37, DI
39  │ │ │ │Dqq                                    33, DE (34)
40  │ │ │ │ │DDsrDsr                              H
41  │ │ │ │ │q                                    10, R
42  │ │ │ │ │Dqq                                  39, R
43  │ │ │ │DDDsrDsrDDsrDsr                        40, 41, 42, DI
44  │ │ │DDDsrDsrDDsrDsr                          34–38, 39–43, DE
45  │ │ │Dsr                                      44, DCF
46  │ │Dsr                                        30–32–45, DE
47  │ │Dss (See steps 26–36 in the proof of (2).) 11, 46
48  │DDDDpDqrDDDsrDDpsDpsDpDpqDDp-
    │ DqrDDDsrDDpsDpsDpDpqDDDpDqr-
    │ DDDsrDDpsDpsDpDpqDDpDqrDDDsr-
    │ DDpsDpsDpDpq                              1, 29, 47, DI
49  │DDpDqrDDDsrDDpsDpsDpDpq.                   48, DCF
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Therefore the proofs are all completed.

References

[1] K. Iséki: Symbolic Logic I—Propositional Calculi (in Japanese). Maki Publisher (1968).

[2] T. W. Scharle: Axiomatization of propositional calculus with Sheffer functors. Notre Dame Jour. of Formal Logic, **6** (1965).

[3] D. F. Siemens: An extension of "Fitch's Rules". Zeitschr. f. math. Logik und Grundlagen d. Math., **7**, 199–204 (1961).