224. *Another Proof of Generalized Sum Theorem for Whitehead Torsion*

By Masami HOSOBUCHI

(Comm. by Kinjirô KUNUGI, M. J. A., Dec. 12, 1968)

Introduction. Kwun and Szczarba [3] established a sum theorem for Whitehead torsion of homotopy equivalence. Their theorem is stated as follows.

Sum Theorem. *Let $f: X \to Y$ be the sum of cellular maps $f_1: X_1 \to Y_1$ and $f_2: X_2 \to Y_2$, where $X = X_1 \cup X_2$, and $Y = Y_1 \cup Y_2$ are finite cell complexes. Suppose that $X_0 = X_1 \cap X_2$ is 1-connected, and that f_1, f_2 and $f_0 = f_1 | X_0 (= f_2 | X_0)$ are homotopy equivalences. Then f is a homotopy equivalence and*

$$\tau(f) = j_{1*}\tau(f_1) + j_{2*}\tau(f_2),$$

where $j_{i*}: \mathrm{Wh}(\pi_1 Y_i) \to \mathrm{Wh}(\pi_1 Y)$ $(i=1, 2)$.

Recently, Hosokawa [1] has generalized this theorem to the case of X_0 being non-simply connected. Indeed, for this general case he obtained the following equality

$$\tau(f) = j_{1*}\tau(f_1) + j_{2*}\tau(f_2) - j_{0*}\tau(f_0),$$

where $j_{0*}: \mathrm{Wh}(\pi_1 Y_0) \to \mathrm{Wh}(\pi_1 Y)$. His proof is accomplished by a geometric idea.

The purpose of this paper is to extend the definition of Whitehead torsion and to prove the generalized sum theorem by making use of the torsions in the extended sense.

For details on the notions of Whitehead group and torsion, we refer the reader to Whitehead [5] and Milnor [4].

§ 1. Let (K, L) be a pair consisting of a finite, connected CW-complex K, and a subcomplex L which is a deformation retract of K. The torsion $\tau(K, L)$ of the pair (K, L) is defined as an element of the Whitehead group $\mathrm{Wh}(\pi_1 K)$, using the chain complex of the universal covering complex of (K, L) [4].

Now let G be a normal subgroup of $\Pi = \pi_1 K$. Then there exists a regular covering $p: (\tilde{K}, \tilde{L}) \to (K, L)$ such that $p_*(\pi_1 \tilde{K}) = G$. $\Pi_G = \pi / G$ operates freely on $(\tilde{K}, \tilde{L})$, and therefore $C_q(\tilde{K}, \tilde{L})$ is a free $Z[\Pi_G]$-module. Since $\tilde{L}$ is a deformation retract of $\tilde{K}$, the chain complex $C(\tilde{K}, \tilde{L})$ is acyclic.

Let $e_1, \cdots, e_\alpha$ denote the q-cells of $K - L$. For each i, choose a representative cell $\tilde{e}_i$ of $\tilde{K}$ lying over e_i. Then $c_q = (\tilde{e}_1, \cdots, \tilde{e}_\alpha)$ can be considered as a basis for the $Z[\Pi_G]$-module $C_q(\tilde{K}, \tilde{L})$. By these preferred bases, the torsion $\tau C(\tilde{K}, \tilde{L}) \in \mathrm{Wh}(\Pi_G)$ can be defined. It is easy to see that this torsion does not depend on the choice of the representative cells $\tilde{e}_i$.

Definition. $\tau C(\tilde{K}, \tilde{L}) \in \mathrm{Wh}(\Pi_G)$ will be called *the torsion of a pair* (K, L) *relative to* G, and denoted by $\tau_G(K, L)$.

Note that $\tau_G(K, L) = \tau(K, L)$ when G is unit.

Now, we shall prove the following theorem.

Theorem 1. *Let* (K, L) *be as above. And let* $p : (\tilde{K}, \tilde{L}) \to (K, L)$, $p' : (\tilde{K}', \tilde{L}') \to (K, L)$ *be regular coverings corresponding to normal subgroups* G, G' *of* $\Pi = \pi_1 K$ *respectively. Furthermore we assume* $G \subset G'$, *and let* $g_* : \mathrm{Wh}(\Pi_G) \to \mathrm{Wh}(\Pi_{G'})$ *be the homomorphism induced by the natural homomorphism* $g : \Pi_G \to \Pi_{G'}$. *Then we have*

$$g_* \tau_G(K, L) = \tau_{G'}(K, L).$$

Proof. Since $p_*(\pi_1 \tilde{K}) = G \subset G' = p'_*(\pi_1 \tilde{K}')$, there exists a (regular) covering map $\tilde{p} : (\tilde{K}, \tilde{L}) \to (\tilde{K}', \tilde{L}')$ such that $p' \circ \tilde{p} = p$ [2]. Hence, for the q-cells $e_1, \cdots, e_\alpha$ of $K - L$, there exist

$$Z[\Pi_G]\text{-basis } (\tilde{e}_1, \cdots, \tilde{e}_\alpha) \text{ for } C_q(\tilde{K}, \tilde{L})$$

and

$$Z[\Pi_{G'}]\text{-basis } (\tilde{e}'_1, \cdots, \tilde{e}'_\alpha) \text{ for } C_q(\tilde{K}', \tilde{L}')$$

such that $\tilde{p}(\tilde{e}_i) = \tilde{e}'_i$ $(i = 1, \cdots, \alpha)$ hold.

By the definition of torsion, it is obvious that the matrices involved in the computation of $\tau_G(K, L)$ corresponds to the matrices involved in computing $\tau_{G'}(K, L)$ under g_*. This completes the proof.

§2. In this section, we shall prove two propositions concerning covering spaces.

Let $p : E \to B$ be a covering space, where E is connected and B is connected, locally pathwise connected.

Proposition 1. *Let* C *be a connected, locally pathwise connected subspace of* B, *and* F *be a component in* $p^{-1}(C)$. *Then the restriction* $\bar{p} = p | F : F \to C$ *is a covering map.*

Proof. For each point $a \in C$, there is a connected open neighbourhood V in B which is covered evenly with respect to p. Put $p^{-1}(V) = \cup V_i$, and $p : V_i \approx V$. Let us take V sufficiently small. Then $U = V \cap C$ is covered evenly with respect to $\bar{p}$ since

$$p^{-1}(U) = (\cup V_i) \cap p^{-1}(C) = \cup U_i,$$

where $U_i = V_i \cap p^{-1}(C)$.

Using these connected neighbourhoods U, U_i, we can show that $\bar{p}(F)$ and $C - \bar{p}(F)$ both are open sets in C. By connectedness of C, we have $\bar{p}(F) = C$.

In particular, let $p : E \to B$ be the universal covering. Then we have:

Proposition 2. $\pi_1 F$ *is isomorphic with* $\mathrm{Ker}(i_* : \pi_1 C \to \pi_1 B)$ *under* $\bar{p}_*$. *Hence* $\bar{p}$ *is a regular covering map.*

Proof. It is well known that p_* is a monomorphism (cf. [2]). We shall consider the following commutative diagram of homotopy exact sequences of pairs.

$$\begin{array}{ccccccccc}
\longrightarrow & \pi_2(E,F) & \xrightarrow{\partial} & \pi_1 F & \xrightarrow{j_*} & \pi_1 E & \longrightarrow \\
& \downarrow{\scriptstyle \hat{p}_*} & & \downarrow{\scriptstyle \bar{p}_*} & & \downarrow{\scriptstyle p_*} & \\
\longrightarrow & \pi_2(B,C) & \xrightarrow{\partial} & \pi_1 C & \xrightarrow{i_*} & \pi_1 B & \longrightarrow
\end{array}$$

Since $\pi_1 E=0$, we have $i_*\bar{p}_*=p_*j_*=0$. Therefore $\mathrm{Im}\,(\bar{p}_*)\subset \mathrm{Ker}\,(i_*)$.

By the lifting theorem [2] we can obtain that $\hat{p}_*$ is onto. Using exactness and commutativity of this diagram, we have that $\bar{p}_*:\pi_1 F \to \mathrm{Ker}\,(i_*)$ is onto.

Corollary 1 (Hosokawa [1, Lemma 2]). *If the map i_* is a monomorphism, then F is the universal covering space over C.*

§ 3. Let (K,L) be a pair of CW-complexes and $(M,N)\subset(K,L)$ be a pair of subcomplexes such that L and N are deformation retracts of K and M respectively, and let $p:(\hat{K},\hat{L})\to(K,L)$ be the universal covering. Then we have an acyclic chain complex $C(p^{-1}(M),p^{-1}(N))$ consisting of free $Z[\pi_1 K]$-modules.

We take a basis of $C(p^{-1}(M),p^{-1}(N))$ as in § 1 and define the torsion $\tau[C(p^{-1}(M),p^{-1}(N))]\in \mathrm{Wh}\,(\pi_1 K)$.

The following is the main theorem in this paper and a generalization of the consequence in the proof of sum theorem [3].

Theorem 2. *Let $k_*:\mathrm{Wh}\,(\pi_1 M)\to \mathrm{Wh}\,(\pi_1 K)$ be the homomorphism induced by the natural homomorphism $k:\pi_1 M\to\pi_1 K$. Then we have*

$$k_*\tau(M,N)=\tau[C(p^{-1}(M),p^{-1}(N))].$$

Proof. Let $(\tilde{M},\tilde{N})$ be a pair of subcomplexes of $(p^{-1}(M),p^{-1}(N))$ consisting of a component $\tilde{N}$ in $p^{-1}(N)$ and a component $\tilde{M}$ in $p^{-1}(M)$ which contains $\tilde{N}$. By Propositions 1 and 2, $\tilde{p}=p\,|\,(\tilde{M},\tilde{N}):(\tilde{M},\tilde{N})\to(M,N)$ is a regular covering. Put $G=\tilde{p}_*(\pi_1\tilde{M})\subset\pi_1 M$.

Since $\mathrm{Ker}\,(k)=G$ we can obtain the monomorphism $h:\pi_1 M/G\to\pi_1 K$. By choosing representative cells in $p^{-1}(M)$ from the single component $\tilde{M}$, we have $h_*\tau_G(M,N)=\tau[C(p^{-1}(M),p^{-1}(N))]$, where $h_*:\mathrm{Wh}\,(\pi_1 M/G)\to\mathrm{Wh}\,(\pi_1 K)$ is induced by h.

On the other hand, it follows from Theorem 1 that $g_*\tau(M,N)=\tau_G(M,N)$, where $g_*:\ \mathrm{Wh}\,(\pi_1 M)\to\mathrm{Wh}\,(\pi_1 M/G)$. Hence $k_*\tau(M,N)=h_*g_*\tau(M,N)=h_*\tau_G(M,N)=\tau[C(p^{-1}(M),p^{-1}(N))]$.

Corollary 2. *Let (K,L) be a pair which is the sum of pairs (K_1,L_1) and (K_2,L_2), where (K_i,L_i) $(i=1,2)$ be the subcomplexes of (K,L). And we put $K_0=K_1\cap K_2$, $L_0=L_1\cap L_2$. Suppose that each L_i is a deformation retract of K_i. Then L is a deformation retract of K and*

$$(*) \qquad \tau(K, L)=j_{1*}\tau(K_1, L_1)+j_{2*}\tau(K_2, L_2)-j_{0*}\tau(K_0, L_0),$$

where $j_{i*}: \mathrm{Wh}(\pi_1 K_i) \rightarrow \mathrm{Wh}(\pi_1 K)$ $(i=0,1,2)$.

Remark. It is clear that this result is equivalent to the generalized sum theorem for the torsion of homotopy equivalences [1].

Proof. We first consider the short exact sequence of free $Z[\pi_1 K]$-modules as in [3]

$$0 \rightarrow C(p^{-1}(K_0), p^{-1}(L_0)) \rightarrow C(p^{-1}(K_1), p^{-1}(L_1)) \oplus C(p^{-1}(K_2), p^{-1}(L_2)) \rightarrow C(\hat{K}, \hat{L}) \rightarrow 0,$$

where $p:(\hat{K}, \hat{L}) \rightarrow (K, L)$ is the universal covering. And the torsion of the middle term is the sum of the torsions of the two extreme terms. Thus

$$\begin{aligned} \tau C(\hat{K}, \hat{L}) &= \tau[C(p^{-1}(K_1), p^{-1}(L_1)) \oplus C(p^{-1}(K_2), p^{-1}(L_2))] \\ &\qquad - \tau C(p^{-1}(K_0), p^{-1}(L_0)) \\ &= \tau C(p^{-1}(K_1), p^{-1}(L_1)) + \tau C(p^{-1}(K_2), p^{-1}(L_2)) \\ &\qquad - \tau C(p^{-1}(K_0), p^{-1}(L_0)). \end{aligned}$$

By Theorem 2 we have $\tau C(p^{-1}(K_i), p^{-1}(L_i)) = j_{i*}\tau(K_i, L_i)$. Hence the equality $(*)$ holds.

References

[1] H. Hosokawa: Generalized product and sum theorems for Whitehead torsion (to appear).

[2] S. T. Hu: Homotopy Theory. Academic Press, New York (1959).

[3] K. W. Kwun and R. H. Szczarba: Product and sum theorems for Whitehead torsion. Ann. of Math., **82**, 183–190 (1965).

[4] J. Milnor: Whitehead torsion. Bull. Amer. Math. Soc., **72**, 358–426 (1966).

[5] J. H. C. Whitehead: Simple homotopy types. Amer. Jour. Math., **72**, 1–57 (1950).