223. On a Product Theorem in Dimension*)

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(Comm. by Kinjirô KUNUGI, M. J. A., Dec. 12, 1968)

1. Let X be a topological space and G an abelian group. The *cohomological dimension* $D(X:G)$ *of* X *with respect to* G is the largest integer n such that $H^n(X, A:G)\neq 0$ for some closed set A of X, where H^* is the Čech cohomology group based on the system of all locally finite open coverings. If X is normal and $\dim X<\infty$, then $D(X:Z)=\dim X$ by [2] and [5, II]. Here $\dim X$ is the covering dimension of X and Z is the additive group of integers.

In this paper we shall show a product theorem for cohomological dimension with respect to certain abelian groups. The theorem is given by proving a product theorem for covering dimension and by applying the same method as developed in [3] and [4]. We use the following groups:

Q=the rational field, Z_p=the cyclic group of order p,

R_p=the subgroup of Q consisting of all rationals whose denominators are coprime with p.

Here p is a prime. Let G be one of the groups Z, Q, R_p, and Z_p, p a prime. We shall show that the relation

$$(*)\qquad D(X\times Y:G)\leqq D(X:G)+D(Y:G)$$

holds if either (i) X is a paracompact Morita space and Y metrizable, or (ii) X is a Lindelöf Morita space and Y a σ-space. See 2 for definition of Morita spaces and σ-spaces. It is well known that the relation $(*)$ is not true for arbitrary groups. Also, the equality $D(X\times Y:G)=D(X:G)+D(Y:G)$ does not generally hold even if G is Q or Z_p, and X and Y are separable metric spaces. Next, let βX be the Stone-Čech compactification of X. If G is finitely generated, then it is known by [5] that $D(\beta X:G)=D(X:G)$. We shall prove that $D(\beta X:G)\geqq D(X:G)$ if X is a paracompact Morita space and G is Q or R_p, p a prime. Throughout the paper all spaces are Hausdorff and maps are continuous.

2. Let $\mathfrak{m}$ be a cardinal number $\geqq 1$. A topological space X is called an $\mathfrak{m}$-*Morita space* if for a set Ω of power $\mathfrak{m}$ and for any family $\{G(\alpha_1,\cdots,\alpha_i)\mid \alpha_1,\cdots,\alpha_i\in\Omega;\ i=1,2,\cdots\}$ of open sets of X such that $G(\alpha_1,\cdots,\alpha_i)\subset G(\alpha_1,\cdots,\alpha_i,\alpha_{i+1})$ for $\alpha_1,\cdots,\alpha_i,\alpha_{i+1}\in\Omega$, $i=1,2,\cdots$,

*) Dedicated to Professor A. Komatsu on his sixtieth birthday.

there is a family $\{F(\alpha_1, \cdots, \alpha_i) | \alpha_1, \cdots, \alpha_i \in \Omega ; i=1, 2, \cdots\}$ of closed sets of X satisfying the following conditions;

(i) $F(\alpha_1, \cdots, \alpha_i) \subset G(\alpha_1, \cdots, \alpha_i)$ for $\alpha_1, \cdots, \alpha_i \in \Omega$, $i=1, 2, \cdots$;

(ii) if $X = \bigcup_{i=1}^{\infty} G(\alpha_1, \cdots, \alpha_i)$ for a sequence $\{\alpha_i\}$ then $X = \bigcup_{i=1}^{\infty} F(\alpha_1, \cdots, \alpha_i)$.

If X is an $\mathfrak{m}$-Morita space for any cardinal number $\mathfrak{m}$, then X is called a *Morita-space.* An $\mathfrak{m}$-Morita space was introduced by Morita [11] and called a $P(\mathfrak{m})$ space, and it played a very important role in the theory of product spaces. A family $\mathfrak{F}$ of subsets of a space is called a *net* if for any point x and any neighborhood U of x there is a member F of $\mathfrak{F}$ such that $x \in F \subset U$. A space X is called a *σ-space* if it is collectionwise normal and it has a σ-locally finite net. Obviously a metrizable space is a σ-space but a σ-space is not necessarily metrizable. Also, it is known that a σ-space is paracompact and perfectly normal (Okuyama [13]). The main theorem in the paper is now stated; its proof is given in the sequence of lemmas.

Theorem 1. *Let G be one of the groups Z, Q, R_p, and Z_p, p a prime. and let X and Y be spaces with finite covering dimension. If either*

(1) *X is a Lindelöf Morita space and Y is a σ-space, or*

(2) *X is a paracompact $\mathfrak{m}$-Morita space and Y is a metrizable space of weight $\leqq \mathfrak{m}$,*

then the following relation holds:

$$(*) \qquad D(X \times Y : G) \leqq D(X : G) + D(Y : G).$$

Let us begin to prove the following lemma.

Lemma 1. *Let X be a normal space and Y a subspace. For any finite open covering $\mathfrak{U}$ of Y, suppose that there is a finite collection $\mathfrak{V}$ of open sets in X such that* (i) *the restriction $\mathfrak{V}|Y$ is a covering of Y which refines $\mathfrak{U}$ and* (ii) *each member of $\mathfrak{V}$ is an F_σ set in X. Then Y is normal and* $\dim Y \leqq \dim X$. *Moreover, if* $\dim X < \infty$ *and G is finitely generated, then $D(Y : G) \leqq D(X : G)$.*

Proof. For a given finite open covering $\mathfrak{U}$ of Y, take a finite collection $\mathfrak{V}$ of open sets of X satisfying the conditions (i) and (ii). Put $X_0 = \cup \{V : V \in \mathfrak{V}\}$. Since X_0 is an F_σ open set of X, X_0 is normal and $\dim X_0 \leqq \dim X$ by [9, Theorem 2.1]. Take a finite open covering $\mathfrak{W}$ of X_0 such that $\mathfrak{W}$ refines $\mathfrak{V}$ and order of $\mathfrak{W} \leqq \dim X + 1$. Since X_0 is normal, $\mathfrak{W}$ is a normal covering. Thus the restriction $\mathfrak{W}|Y$ is a normal covering refining $\mathfrak{U}$ and of order $\leqq \dim X + 1$. This implies that Y is normal and $\dim Y \leqq \dim X$. The second part of the lemma is proved by a similar way as in the proof of [6, Theorem 1].

Lemma 2. *Under the assumption* (1) *or* (2) *in Theorem* 1 $\dim (X \times Y) \leqq \dim (\beta X \times Y)$.

Proof. We only prove the case (1). The case (2) is a consequence of Lemma 1 and [6, Lemma 4]. Let Y be a σ-space and let $\mathfrak{B}=\cup\mathfrak{B}_i$ be a net of Y such that $\mathfrak{B}_i=\{V_{i\alpha}:\alpha\in\Omega\}$, $i=1,2,\cdots$, is a σ-locally finite collection of closed sets. We can assume each $\mathfrak{B}_i$ is closed with respect to finite intersection. Let us put $F(\alpha_1,\cdots,\alpha_i)=\bigcap_{\nu=1}^{i}V_{\nu\alpha_\nu}$, $\alpha_1,\cdots,\alpha_\nu\in\Omega$. Then $\mathfrak{F}_i=\{F(\alpha_1,\cdots,\alpha_i):\alpha_1,\cdots,\alpha_i\in\Omega\}$ is locally finite in Y for $i=1,2,\cdots$. Since Y is collectionwise normal and countably paracompact, there is a collection $\mathfrak{W}_i=\{W(\alpha_1,\cdots,\alpha_i):\alpha_1,\cdots,\alpha_i\in\Omega\}$ of open sets of Y such that

(2.1) $F(\alpha_1,\cdots,\alpha_i)\subset W(\alpha_1,\cdots,\alpha_i)$,

(2.2) $\mathfrak{W}_i$ is locally finite in Y for $i=1,2,\cdots$.

Let $\mathfrak{U}=\{U_k:k=1,\cdots,s\}$ be a finite open covering of $X\times Y$. For $k=1,\cdots,s$ and $\alpha_1,\cdots,\alpha_i\in\Omega$, let $T(\alpha_1,\cdots,\alpha_i:k)=\{T_\lambda\}$ be the collection of subsets in X satisfying the following condition;

(2.3) each T_λ is an open F_σ set in X and there is an open set V_λ in Y such that $F(\alpha_1,\cdots,\alpha_i)\subset V_\lambda\subset W(\alpha_1,\cdots,\alpha_i)$ and $T_\lambda\times V_\lambda\subset U_k$.

Put $T(\alpha_1,\cdots,\alpha_i:k)=\cup\{T_\lambda:T_\lambda\in\mathfrak{T}(\alpha_1,\cdots,\alpha_i:k)\}$ and $T(\alpha_1,\cdots,\alpha_i)=\bigcup_{k=1}^{s}T(\alpha_1,\cdots,\alpha_i:k)$. Then $T(\alpha_1,\cdots,\alpha_i,\alpha_{i+1})\supset T(\alpha_1,\cdots,\alpha_i)$ for $\alpha_1,\cdots,\alpha_i,\alpha_{i+1}\in\Omega$, $i=1,2,\cdots$, and $\{T(\alpha_1,\cdots,\alpha_i)\times F(\alpha_1,\cdots,\alpha_i):\alpha_1,\cdots,\alpha_i\in\Omega, i=1,2,\cdots\}$ covers $X\times Y$. Since X is a Morita space, there is a collection $\{S(\alpha_1,\cdots,\alpha_i):\alpha_1,\cdots,\alpha_i\in\Omega, i=1,2,\cdots\}$ of closed sets in X such that

(2.4) $S(\alpha_1,\cdots,\alpha_i)\subset T(\alpha_1,\cdots,\alpha_i)$, $\alpha_1,\cdots,\alpha_i\in\Omega$, $i=1,2,\cdots$, and $\{S(\alpha_1,\cdots,\alpha_i)\times F(\alpha_1,\cdots,\alpha_i):\alpha_1,\cdots,\alpha_i\in\Omega, 1,2,\cdots\}$ covers $X\times Y$.

Since $S(\alpha_1,\cdots,\alpha_i)$ is normal and $\{T(\alpha_1,\cdots,\alpha_i:k):k=1,\cdots,s\}$ covers $S(\alpha_1,\cdots,\alpha_i)$, there is a closed set $P(\alpha_1,\cdots,\alpha_i:k)$ in X such that $S(\alpha_1,\cdots,\alpha_i)=\bigcup_{k=1}^{s}P(\alpha_1,\cdots,\alpha_i:k)$ and $P(\alpha_1,\cdots,\alpha_i:k)\subset T(\alpha_1,\cdots,\alpha_i:k)$ for $k=1,\cdots,s$. Now the collection $\mathfrak{T}(\alpha_1,\cdots,\alpha_i:k)$ covers a Lindelöf space $P(\alpha_1,\cdots,\alpha_i:k)$ and hence a countable subcollection $\{T_{\lambda j}:j=1,2,\cdots\}$ of $\mathfrak{T}(\alpha_1,\cdots,\alpha_i:k)$ which covers $P(\alpha_1,\cdots,\alpha_i:k)$. For each member $T_{\lambda j}$, take an open F_σ set $H_{\lambda j}$ in βX such that $H_{\lambda j}\cap X=T_{\lambda j}$. For each $H_{\lambda j}$, there is an open set $V_{\lambda j}$ of Y by (2.3) such that $(H_{\lambda j}\times V_{\lambda j})\cap(X\times Y)\subset U_k$. Put $H(\alpha_1,\cdots,\alpha_i:k)=\bigcup_{j=1}^{\infty}H_{\lambda j}\times V_{\lambda j}$. Then $H(\alpha_1,\cdots,\alpha_i:k)$ is an open F_σ set in $\beta X\times Y$ such that

$$(2.5)\quad \begin{aligned} P(\alpha_1,\cdots,\alpha_i:k)\times F(\alpha_1,\cdots,\alpha_i)&\subset H(\alpha_1,\cdots,\alpha_i:k)\cap(X\times Y)\\ &\subset U_k\cap(S(\alpha_1,\cdots,\alpha_i:k)\times W(\alpha_1,\cdots,\alpha_i)).\end{aligned}$$

Finally, put $V_k^i=\cup\{H(\alpha_1,\cdots,\alpha_i:k):\alpha_1,\cdots,\alpha_i\in\Omega\}$ and $V_k=\bigcup_{i=1}^{\infty}V_k^i$.

Since the collection $\{S(\alpha_1, \cdots, \alpha_i : k)\times W(\alpha_1, \cdots, \alpha_i) : \alpha_1, \cdots, \alpha_i \in \Omega\}$ is locally finite in $\beta X\times Y$ by (2.2), $\{H(\alpha_1, \cdots, \alpha_i : k) : \alpha_1, \cdots, \alpha_i \in \Omega\}$ is locally finite in $\beta X\times Y$. Thus V_k^i and hence V_k are open F_σ in $\beta X\times Y$. Therefore the conditions (i) and (ii) in Lemma 1 are satisfied if $\beta X\times Y$, $X\times Y$, $\mathfrak{U}$, and $\{V_k\}$ replace X, Y, $\mathfrak{U}$, and $\mathfrak{V}$ respectively. The lemma follows from Lemma 1.

Proof of Theorem 1 in case $G=Z$. By Morita [10, Theorem 4] we know $\dim(\beta X\times Y)\leqq \dim \beta X+\dim Y=\dim X+\dim Y$. Thus the theorem is a consequence of Lemma 2.

Nagami [12] showed that if X is a paracompact Morita space and Y is a σ-space then $X\times Y$ is paracompact. It is open whether the relation $\dim(X\times Y)\leqq \dim X+\dim Y$ is true or not in case X is a paracompact Morita space and Y is a σ-space.

Next, let us prove Theorem 1 in case G is either Q, R_p or Z_p, p a prime. We shall apply a technique used in [3, p. 49] and [4, pp. 171–172] and lately by Kuzminov. Consider the 2-dimensional Cantor manifolds M_0, M_p in [3, p. 44] and Pontrjagin's Cantor manifold P_p. We denote M_0, M_p, and P_p by M_Q, M_{R_p}, and M_{Z_p} respectively.

Lemma 3. *Let X be a paracompact space with $\dim X<k$ and let G be any of the groups Q, R_p, and Z_p, p a prime. Then $D(X:G)=\dim(X\times M_G^k)-2k$. Here M_G^k is the k-fold product $M_G\times M_G\times\cdots\times M_G$.*

If $G=Z_p$, then the lemma is proved by Kuzminov [7]. To complete the proof, as known in the proof of [4, Theorem 2], it is enough to show the following lemma.

Lemma 4. *Let $G=Q$ or R_p, p a prime. If X is a paracompact space with finite covering dimension, then*

(1) $D(X:G)=\dim X$ *if and only if* $\dim(X\times M_G)=\dim X+2$.

(2) $$D(X\times M_G:G)=D(X:G)+2.$$

Proof. We prove only (2). The proof of (1) is similar. Let us remind the construction of M_G. Let T be the boundary of M_G (see [3, p. 44]) and let $\tilde{M}_G=M_G/T$ and t_0 the point corresponding to T. Then it is easy to show that

$$H^2(\tilde{M}_G)\cong G,\ H^1(\tilde{M}_G)=0,\quad\text{and}\quad H^0(\tilde{M}_G)\cong Z. \tag{2.6}$$

Let A be a closed set of X and let $\tilde{X}=X/A$ and a_0 the point corresponding to A. Let $n>0$. Then we have

$$\begin{aligned}&H^{n+2}((X, A)\times(M_G, T):G)\cong H^{n+2}((\tilde{X}, a_0)\times(\tilde{M}_G, t_0):G)\\ &\cong H^{n+2}(\tilde{X}\times\tilde{M}_G:G)\cong H^n(\tilde{X}:G)\oplus H^{n+2}(\tilde{X}:G)\cong H^n(X, A:G)\\ &\oplus H^{n+2}(X, A:G).\end{aligned}\tag{2.7}$$

The first isomorphism is a consequence of [6, Lemma 6], the second follows from the cohomology sequence of $(\tilde{X}\times\tilde{M}_G, \tilde{X}\times\{t_0\}\cup\{a_0\}\times\tilde{M}_G)$, the third comes from (2.6) and [1, Theorem C], and the fourth is trivial. To complete the proof, let $D(X:G)=n$. We can assume that $n>0$. There is a closed set A of X such that $H^n(X, A:G)\neq 0$. By (2.7) we can know $D(X\times M_G:G)\geqq n+2$. Conversely, let $D(X\times M_G:G)=n+2$. Then, by [5, I, Theorem 5] and the structure of M_G, there is a closed set A of X such that $H^{n+2}((X, A)\times(M_G, T):G)\neq 0$, where T is the boundary of M_G. By (2.7) $H^n(X, A:G)\neq n$ and hence $D(X:G)\geqq n$. This completes the proof.

Proof of Theorem 1 in case G is either Q or R_p. Let k be a positive integer such that $k>\text{Max}(\dim X, \dim Y)$. Since $\dim(X\times Y)<2k$, by the theorem proved already in case $G=Z$, Lemma 3 means that $D(X\times Y:G)=\dim(X\times Y\times M_G^{2k})-4k$. Since M_G^k is a compact metric space, if X is an $\mathfrak{m}$-Morita space then $X\times M_G^k$ is an $\mathfrak{m}$-Morita space by [11, Corollary 3.5] and if Y is a σ-space then $Y\times M_G^k$ is a σ-space. Thus we know that $\dim(X\times Y\times M_G^{2k})\leqq\dim(X\times M_G^k)+\dim(Y\times M_G^k)$. Hence $\dim(X\times Y\times M_G^{2k})-4k\leqq\dim(X\times M_G^k)-2k+\dim(Y\times M_G^k)-2k=D(X:G)+D(Y:G)$ by Lemma 3. This completes the proof.

Let Q_p be the additive group of p-adic rationals mod 1. Then $D(M_{Z_p}\times M_{Z_p}:Q_p)=3$ and $D(M_{Z_p}:Q_p)=1$. Hence Theorem 1 is not generally true for $G=Q_p$. Also, we can not take the equality in place of (*) in Theorem 1 even if G is Q or Z_p. Because, let X be the set of points in Hilbert space all of whose coordinates are rational. Since $\dim X=1$ and $X\times X$ is homeomorphic to X, $D(X:G)=D(X\times X:G)=1$ for any group G.

Theorem 2. *Let $G=Q$ or R_p, p a prime. If X is a paracompact 2-Morita space with finite covering dimension, then $D(\beta X:G)\geqq D(X:G)$, where βX is the Stone-Čech compactification of X.*

Proof. Let $\dim X<k$. By [11, Corollary 4.6] X is an $\aleph_0$-Morita space. Since weight of $M_G^k=\aleph_0$, Lemma 2 shows that $\dim(X\times M_G^k)\leqq\dim(\beta X\times M_G^k)$. Thus $D(X:G)=\dim(X\times M_G^k)-2k\leqq\dim(\beta X\times M_G^k)-2k=D(\beta X:G)$. This completes the proof.

Let X_0 be the metric space constructed by P. Roy [14]. Then $\operatorname{ind} X_0=0$ and $\dim X_0=1$. Take the Freudenthal compactification γX_0 of X_0. Then $\operatorname{ind}\gamma X=0$ by [8, Theorem 6] and hence $\dim\gamma X_0=0$ by the compactness of γX_0. Thus $D(X_0:G)>D(\gamma X_0:G)$ for any group G. Therefore Theorem 2 is not generally true for an arbitrary compactification. The fact mentioned above was informed to me by Professor Morita.

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